Chapter One
Introduction to Engineering Mechanics

"Mechanics is the paradise of mathematical science because here we come to the fruits of mathematics".

Leonardo da Vinci (1452-1519), Italy

Mechanics: Mechanics is a subject that matters to everybody everywhere because it deals with the study of how and why things move and what is equally important, how we can make things move the way we want them for the purposes of our work and our comfort – in fact, to aid us in our daily lives.

Why do bodies move as they do? Why do bodies remain at rest as they do? These questions engaged the attention of philosophers, mathematicians and scientists for many centuries. The branch of physical science which deals with the state of rest or of motion of bodies under the action of forces is called *mechanics.* Mechanics is based on the concepts of time, space, force, energy & matter. Archimedes (287-212 BC), Galileo Galilee (1564-1642), Sir Issac Newton (1624-1727) and Albert Einstein (1878-1955) have contributed a lot to the development of mechanics. Substantial contributions have been made by Pierre Varignon, Euler, D'Alembert and others also.

The mechanics, which is developed as on today, may be grouped as below:

- ▶ Classical (Newtonian) mechanics
- ▶ Relativistic mechanics
- ▶ Quantum or wave mechanics

The mechanics based on Newton's laws of motion is called **Classical/Newtonian mechanics.**

Engineering Mechanics: While scientists are keen on understanding and establishing the laws of mechanics, engineers are interested in applying these laws to the actual field problems. Application of the laws of mechanics to actual problems is termed '*Engineering mechanics*' or '*Applied mechanics*'.

Classification of Engineering Mechanics: Engineering mechanics may be grouped into the following three heads.

1. Mechanics of Rigid Bodies
2. Mechanics of Deformable Bodies
3. Mechanics of Fluids

A body in which all the particles occupy the fixed position with respect to each other is called the **rigid body.** However, no body is rigid; every body undergoes some deformation when acted by forces. Still there exist several situation wherein there is no deformation of the body, the concept of rigid body holds good. The mechanics of rigid bodies dealing with bodies with rest is termed as **'Statics'** and that dealing with the bodies in motion is termed as **'Dynamics'.** The dynamics is further divided into two streams – Kinematics and kinetics. **'Kinematics'** is the branch of dynamics which deals with the motion of bodies without referring to the forces causing the motion of the body and **'Kinetics'** is the branch of dynamics which deals with the motion of bodies referring to the forces causing the motion.

If our interest is to study the internal forces developed in a body when subjected to external forces, deformation of the body should be considered. The mechanics involved in these studies is called **mechanics of deformable bodies. 'Strength of materials', 'Theory of elasticity'** and **'Theory of plasticity'** are some of the titles under which this study is covered.

Mechanics of rigid and deformable bodies together is known as **'Solid Mechanics'.**

A substance which deforms continuously even with the application of a small shear force (force trying to cut the body) is called fluid. Liquids and gases are examples of fluid. The mechanics dealing with these materials is called **Fluid mechanics.** This is covered under the titles 'Mechanics of ideal fluids', 'Mechanics viscous fluids', and 'Mechanics of incompressible fluids'.

This basic course on engineering mechanics deals only with the mechanics of rigid bodies. Summary of classification engineering mechanics is shown below as a flow chart.

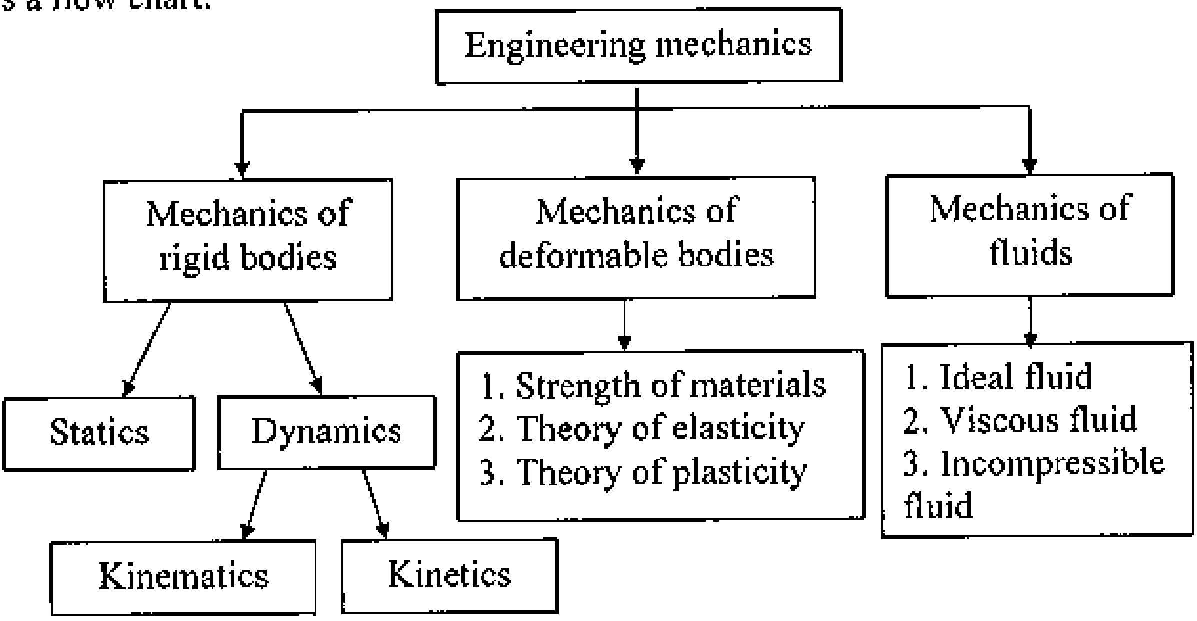


Figure 1: Branches of Mechanics

Problems in Engineering Mechanics: Engineers conceive, plan, design, and construct buildings, machines, airplanes, and countless other objects, for the comfort and use of the human race. Each of these objects servers a definite and useful purpose; behind each lies an absorbingly interesting storey of engineering skill and achievement.

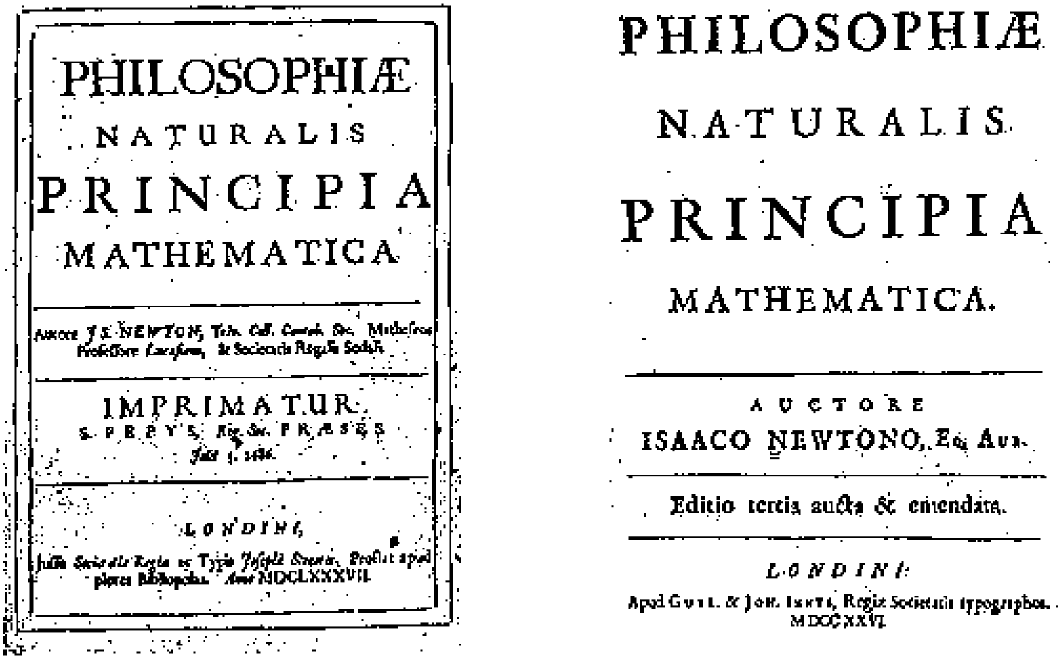
Having in mind a definite purpose for the object to be constructed, the engineer then conceives its appropriate form, either entirely new, or an improvement upon one already in existence. Next, he must analyze and determine the forces, known & unknown, acting on the object, and the motions, if any, of its various related parts.

To do this successfully *the engineer must have a through knowledge and understanding of the principles of mechanics, and of their applications to his particular problem.* Having thus determined the forces and the motions involved, he may proceed with the design of the object, using available materials of suitable strength and other requisite properties. The final size & shape of the object, and each of its separate parts, may then be expressed in blueprints, after which the object is already for production.

From this brief analysis of the work of the engineer, we see clearly that *a knowledge of mechanics is an fundamental to success in the field of engineering as is an understanding of the alphabet to those who would learn to read and write their own language.* The extent of his knowledge of mechanics may have an important bearing on the opportunities that will open to the student in this great field of work. Certainly without some knowledge and training in mechanics he would have little or no chance of entering the engineering profession.

Fundamental Laws of Mechanics: The following are considered fundamental laws in classical (Newtonian) mechanics:

- Newton's first law
- Newton's second law
- Newton's third law
- Law of transmissibility of forces and
- Parallelogram law of forces



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AUCTORE
ISAACO NEWTONO, Eq. Aur.

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Apud Guil. & Joh. Innys, Regiæ Societatis typographos.
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Figure 2: Cover page of "*Principia Mathematica*" by Sir Issac Newton.

Newton's Laws of Motion: The science of analytical mechanics is based on Newton's laws of motion.

- ▶ *(I) Every particle remains in a state of rest or moves with a constant velocity in a straight line unless an unbalanced force acts on it.*
- ▶ *(II) The acceleration of a particle is directly proportional to the resultant force acting on it and inversely proportional to its mass, and the sense of acceleration is the same as that of the resultant force. (i.e. The rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction in which the force acts.)*

$$F \infty \frac{d}{dt}(mv)$$

$$\Rightarrow F \infty m\frac{dv}{dt}$$

$$\Rightarrow F \infty ma$$

$$\Rightarrow F = kma \quad [\text{k is constant of proportionality}]$$

It should be possible to obtain from this expression the concept of a unit of force. The unit of force which at once suggests itself is that which would produce a unit of acceleration in a unit of mass. If we accept this definition of the unit of force and substitute in the fundamental formula F = kma, F = 1, m = 1 & a = 1, we see that k = 1, irrespecting of whether we are dealing with C.G.S. or F.P.S. or M.K.S. units.

$$\Rightarrow F = ma$$

provided only that we are dealing in the appropriate units of force, mass & acceleration.

▶ *(III) To every action, there is an equal and opposite reaction.*

If a body 'A' exerts force F_{AB} on another body 'B' then body 'B' also exerts force F_{BA} on body 'A'.

$$F_{AB} = -F_{BA}$$

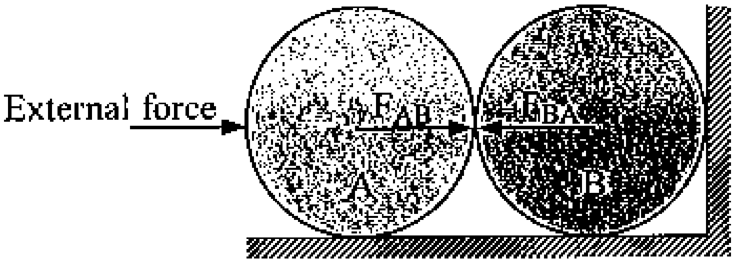


Figure 3: Newton's third law

Derivation of first law from the second law: $\Rightarrow F = ma$

$$\Rightarrow F = m\frac{(v_2 - v_1)}{t}$$

if no external force is applied, F = 0

$$\Rightarrow m\frac{(v_2 - v_1)}{t} = 0 \qquad [m \neq 0]$$

$$\Rightarrow v_2 = v_1$$

$\Rightarrow$ final velocity = initial velocity (still body remains at rest or in constant motion)

Scalar and Vector Quantities: A quantity is said to be scalar if it is completely defined by its magnitude alone. Examples: length, area, volume, time.

A quantity is said to be vector if it is completely defined only when its magnitude and direction are specified. They obey parallelogram law of addition. Example: Force, displacement, velocity, acceleration.

Body & Particle: A body is a material configuration in space. The matter in it possesses a definite mass.

A particle is a geometric point possessing constant mass.

Classification of Concepts: Five classes of concepts forming the theoretical basis of mechanics are:

Undefined terms: The terms whose definitions are inadequate or not available are called the undefined terms.

Defined terms: The terms defined by the undefined terms are called defined terms. Consequently, their definition may be inadequate.

Axioms (Postulates): The assertions based on experience or observations and which can be admitted without proof are called axioms (postulates).

Assumptions (Hypotheses): The simplifications based on experiments or observations and which can be admitted within certain limits are called assumptions (hypotheses).

Laws (Principles): The statements (conclusions) proven by analytical reasoning or based on experimental evidence are called laws (principles).

Basic Concepts: Three typical undefined terms, called basic concepts of mechanics, are space, time & force. They can be interpreted and/or measured experimentally.

Space: The region in which the event (rest or motion) takes place is called space. The measurement of space is length* and in engineering the space is three dimensional.

Time: The measure of succession of events related to an arbitrarily selected datum (start, beginning) is called datum. Static quantities are assumed to be independent of time**.

Force: The action of one body on another, interpreted as push or pull, is called force.

[*length – a concept for describing size quantitatively & **time – a concept for ordering the flow of events.]

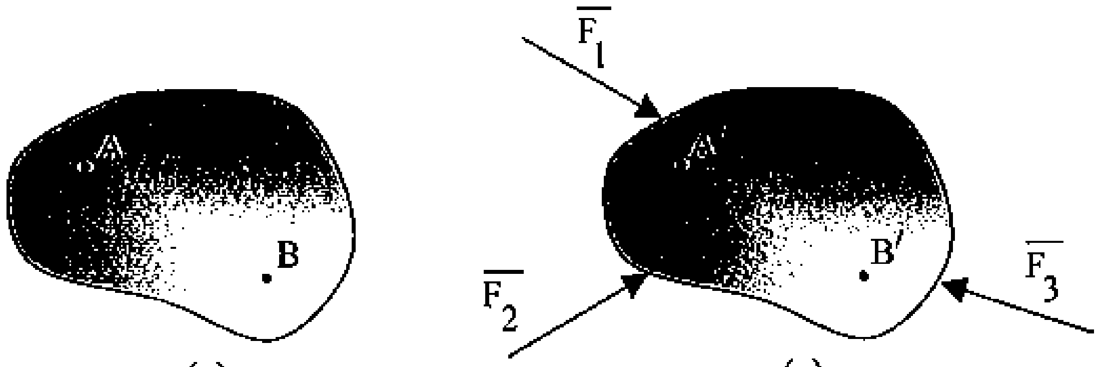
Associated Concepts: In addition to the basic concepts, three associated concepts are necessary in the analysis: particles, rigid body & configuration.

Idealizations in Mechanics: A number of ideal conditions are assumed to exist while applying the laws of mechanics to practical problems. In fact, without such assumptions, it is not feasible to arrive at practical solutions. The experience in the past centuries has shown that the following idealization could be made as they do not bring down the accuracy of the analytical results below the optimum level required by engineers to deal with practical problems.

Continuum: A body consists of several particles. It is well known fact that each particle can be sub-divided into molecules, atoms and electrons. It is not feasible to solve any engineering problems by treating a body as a conglomerate of such discrete particles. The body is assumed to consist of a continuous distribution of matter which will not separate even when various forces considered are acting simultaneously. In other words, we say the body is treated as a continuum.

Rigid Body: A rigid body may be defined as a body in which the relative position of any two particles does not change even after the application of a system of forces. For example, let the body as shown in figure 4(a) move to a position as shown in figure 4(b) when the system of forces $\bar{F}_1$, $\bar{F}_2$ & $\bar{F}_3$ are applied. If the body is treated as rigid body, the relative position of A to B is the same as $A^/$ and $B^/$.

$$AB = A^/B^/$$



(a) (a)

Figure 4: Rigid body

Particle: A particle may be defined as an object which has only mass and no size. Theoretically, such a body can not exist; while dealing with the problems involving distances which are considerably larger compared to the size of the body, the size of the body may be neglected without sacrificing the accuracy. The examples are as follows:

- A bomber aero-plane is a particle for a gunner operating from the ground.
- A ship in the mid sea is a particle in the study of relative motion from the control tower in a port.
- In the study of the movement of earth in celestial space, earth is treated as particle.

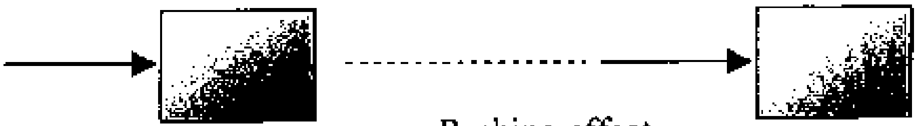
Configuration: The arrangement of particles in a system (in a body) is called configuration. All static configurations independent of time.

Point Force: It is yet another idealization which is very commonly used in engineering mechanics. In figure 7, the weight of the man standing on the ladder is shown as a force applied at the point C. Actually the man can not apply his weight through a single point. There is a certain area of contact through which the weight is applied. However, this area is small compared to the other dimensions involved in the problem. Therefore, not much accuracy is lost by treating it as a point force and thereby simplifying the problem.

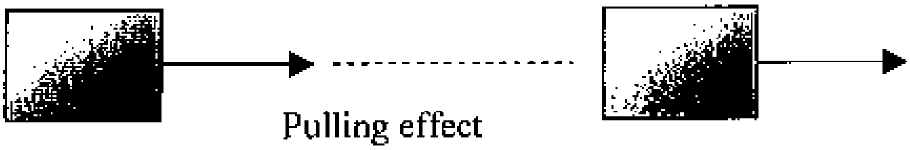
Force: It is the action of a body on the other body. It always tends to change the state of rest or state of motion of the other body. It is associated with direction also.

Force may be defined as an agent which produces or tends to produce, destroys or tends to destroy motion.

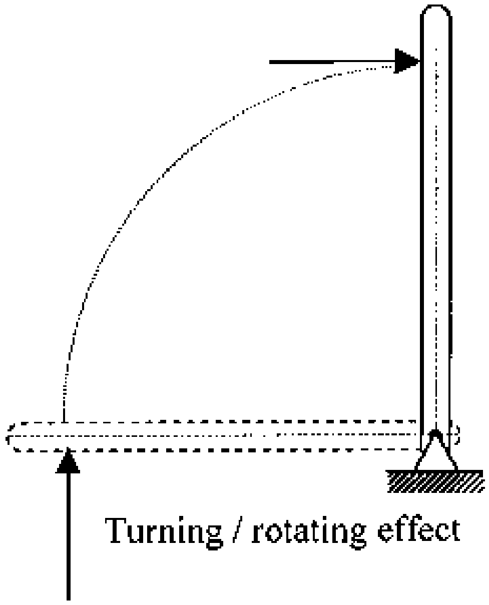
A force can push or pull a rigid body or/and can cause a rigid body to turn around or rotate. Pulling force is called tensile force/positive force & pushing force is called compressive force/negative force.



Pushing effect



Pulling effect



Turning / rotating effect

Figure 5: Effect of force application (translation or/and rotation)

Defining of a Force: A force could be completely defined only when the following four characteristics are specified.

- ▶Magnitude
- ▶Line of action
- ▶Direction
- ▶Point of application

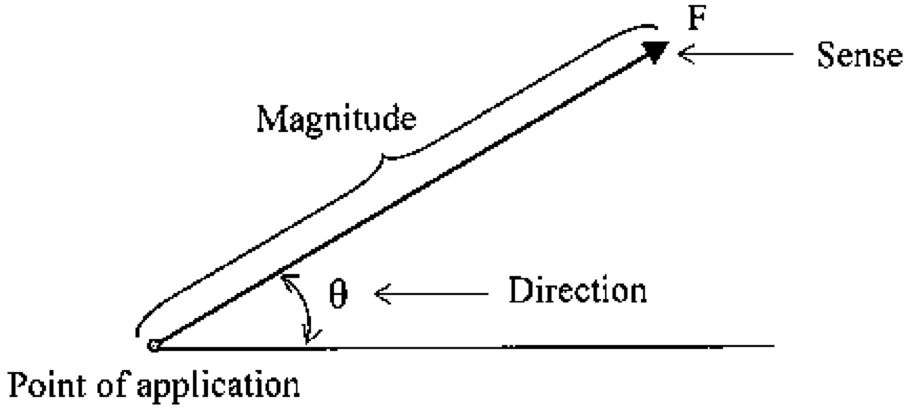


Figure 6: Characteristics of a force

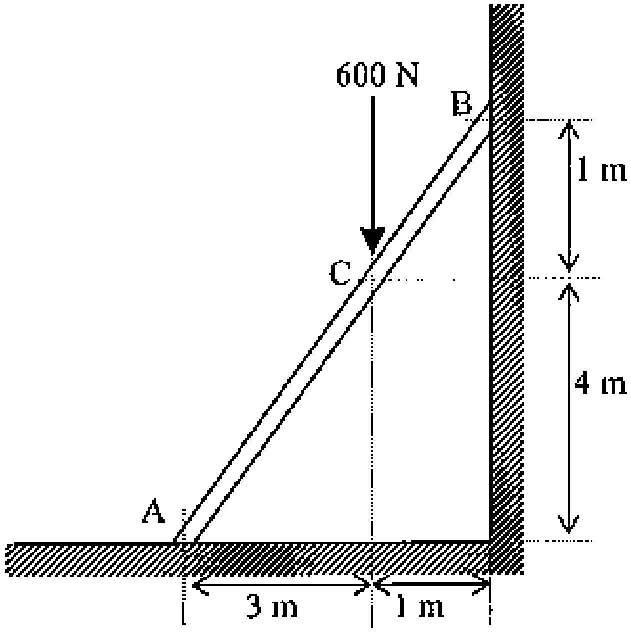


Figure 7

In figure 7, AB is a ladder kept against a wall. At point C, a person weighing 600 Newton is standing. The force applied by the person on the ladder has the following characteristics:

- Magnitude is 600 Newton
- The line of action is vertical
- It's direction is downward
- The point of application is at C

The magnitude of the force is written near the arrow. The line of arrow shows the line of action of the force and the arrow head shows the direction of the force. The point of intersection of the force with the body shows the point of application of the force on the body.

Units of Force: Dyne, Poundal, Newton, Gram-weight, Pound-weight, Kilogram-weight.
Newton is written as N
Pound-weight is written as lb. (comes from Latin word *libra pondo* abbreviated as lb.)
Kilogram-weight is written as kg.
Gram-weight is written as gm.

Force = mass × acceleration
Weight = mass × gravitational acceleration

Absolute system of force units used by physical scientists:
1 Newton = 1 kilo-gram × 1 meter/sec^2
1 Poundal = 1 Pound × 1 ft/sec^2
1 Dyne = 1 gram × 1 cm/sec^2

Gravitational system of force units used by engineers:
1 Kilogram-weight = 1 Kilogram × 9.81 meter/sec^2
1 Pound-weight = 1 Pound × 32.2 ft/sec^2
1 gram-weight = 1 gram × 981 cm/sec^2

What is Weight / is Weight a Force? Everybody possessing mass is pulled down by gravitation force. It is called weight of the body which acts vertically downward. So, weight is a force whose

- ►Magnitude is equal to the weight of the body
- ►The line of action is vertical
- ►It's direction is downward
- ►The point of application is at c.g. (center of gravity)

Mass & Weight: Mass is the quantity of matter in a body & weight is a force – the force of the earth's pull on the body. Weight is concerned with that particular kind of force called the force of gravity. The earth exerts an attracting force on all bodies which come within its sphere of influence. That is why, when we hold an object at some height above the level of the ground and let go, it immediately drops under the action of the force of gravity. We can feel this force of gravity in the weight of bodies. Weight & mass are intimately connected, for the force of gravity, upon which the weight of a body depends, is directly proportional to its mass

$$W \infty m$$
$$\Rightarrow W = mg$$

Where, W = weight, m = mass & g = acceleration due to gravity
Although weight of a body alters with its position, mass is always constant. The quantity of matter in a body is in no way altered by taking it up a mountain or down a mine.

Fundamental Units: The measurement of physical quantities is one of the most important operations in engineering. Every quantity is measured in terms of some arbitrary, but internationally accepted units called **fundamental units.** All physical quantities met with in classical (Newtonian) mechanics are expressed in terms of the following three fundamental quantities.

(1) length (2) mass & (3) time

System of Units: There are three system of unit, which are commonly used and universally accepted. They are known as:

C.G.S. unit: In this system, the fundamental units of length, mass and time are centimeter, gram and second respectively.

F.P.S. unit: In this system, the fundamental units of length, mass and time are foot, pound and second respectively.

M.K.S. unit: Strictly speaking, the fundamental units of this system are not separate ones, but are multiples of C.G.S. units. In this system, the fundamental units of length, mass and time are meter, kilogram and second respectively.

Table 1: Symbols/prefix for multiplication factor

Multiplying factor	Prefix	Symbol
10^{12}	tera	τ
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{0}	----	----
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	η
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

What Happens when Several Forces act on a body at the Same Time? In nature, and in the world of industry, almost always more than one force is acting on a body at the same time. As a result, the forces may be as a whole "balance out" or be "out of balance".

(a) When they exactly balance, the body is:

(i) still if it was still at first;

(ii) moving steadily and uniformly, if is was doing this at first.

(b) When the forces are out of balance, there is an overall force effect on the aircraft and this must produce a change of motion, so the body or machine must move faster or slower than before.

The Two Main Divisions of Mechanics: So we get the two special branches into which the study of Mechanics is divided:

(a) The study of forces that produce balance or equilibrium – this branch of mechanics is called *Statics*.

(b) The study of forces that are out of balance, and so produce movements - this branch of mechanics is called *Dynamics*.

The Principle of Inertia & the Meaning of Force: "no inanimate body can of itself change its own state of rest on of uniform motion in a straight line". – this is known as principle of inertia, and is the basis of whole subject of mechanics. This principle of inertia really has no meaning apart from the idea of a force. It is true that because of its inertia, a body cannot alter its own state of rest or motion. Only an external force applied to the body can do this. Indeed, the full property of inertia tells us that the natural tendency is for all bodies to resist change in their movements. It is just as though they want to go on doing what they have been doing. That is why, for example, when we are cycling on a road with a steady speed, if we put on the brakes too quickly we are quite likely to find ourselves shooting over the handlebars to be thrown on to the road ahead of the bicycle. Our own body's "inertia" wants us to go on doing what we were doing before we applied the brakes, i.e., moving along the road on our bicycle. The sudden braking stops the bicycle but it fails to stop us. Hence, we are thrown forward.

Again, imagine that we are sitting in a train that has stopped at a station, and that it starts moving forward very suddenly. Our own inertia "resists" this change and "wants" us to go on being still on the seat, and the result is that we feel ourselves jerked backwards against the seat if we facing the engine. If we have our backs to the engine we tend to fall off the seat.

If both cases, of course, the manifestation of a body's inertia is only seen when an external force is applied to make the body move if it is still, or to make it move differently if it is already in motion.

Forces & their Effects: We see from the above that one of the effects of a force is to change the state of rest or of motion of a body. This is, however, only one of the possible effects of a force on a body. In the concluding sentence of the paragraph defining the term force, we made use of the phrase "changes or tends to change". In other words, a force does not always succeed in changing a body's state of rest or motion. Whether it succeeds in doing this depends entirely upon whether the body is perfectly free to move or not.

We may classify all bodies from this point of view as being:
(a) perfectly free to move;
(b) perfectly restrained, i.e., completely unable to move;
(c) partially restrained, i.e., partially able to move.
As an example of (a) we may quote any body in space, such as a star, or cricket-ball in the air, or a stone dropped from a tower; as an example of (b0 a pillar or column reaching from floor to balcony in a hall, or a beam or girder supporting a roof; as an example of (c) a door swinging on its hinges, or a pendulum suspended from appoint, or an electric light hanging from a ceiling.

It is important to notice that the effect of a force on a body depends entirely upon which of these three conditions the body happens to be in.
(a) If the body is perfectly free to move, the effect of a force will be to move it if it is at rest, or to change the nature of its motion if it is already moving.
(b) If the body is completely unable to move, the effect of a force will be to set up what we call a stress in the body (this is a part of solid mechanics).

(c) Finally, if the body is partially able to move, the effect of the force will be to set up a swing or rotation about the point or line of restraint.

The whole of the science of mechanics is, in fact, a study of these three separate conditions and their consequences.

Classification of Force Systems / System of Forces: When several forces act upon a body simultaneously, they constitute a system of forces. Various system of forces, their characteristics and examples are given in table 2 below.

Table 2: Characteristics of various systems of forces

Force system	Characteristics	Examples
Collinear forces	Line of action of all forces act along the same line	The forces on a rope in a tug of war
Coplanar parallel forces	All the forces are parallel to each other and lie in a single plane	The system forces acting on a beam subjected to a vertical loads (including reactions)
Coplanar- like parallel forces	All the forces are parallel to each other, lie in a single plane and act in the same direction	The weight of a stationary train on rail when the track is straight
Coplanar concurrent forces	Line of action of all the forces pass through a single point and the forces lie in the same plane	The forces on a rod resting against a wall
Coplanar non-concurrent forces	All the forces don't meet at a point, but lie in a single plane	The forces on a ladder resting against a wall when a person stands on a rung which is not at its center of gravity .
Non-coplanar parallel forces	All the forces that are parallel to each other, but not in the same plane.	The weight of benches in a class room
Non-coplanar concurrent forces	All the forces do not lie in the same plane but their lines of action pass through a single point.	A tripod carrying a camera

Force system	Characteristics	Examples
Non -coplanar non-concurrent forces	All the forces do not lie in the same plane and their lines of action do not pass through a single point.	The forces acting on a moving bus.

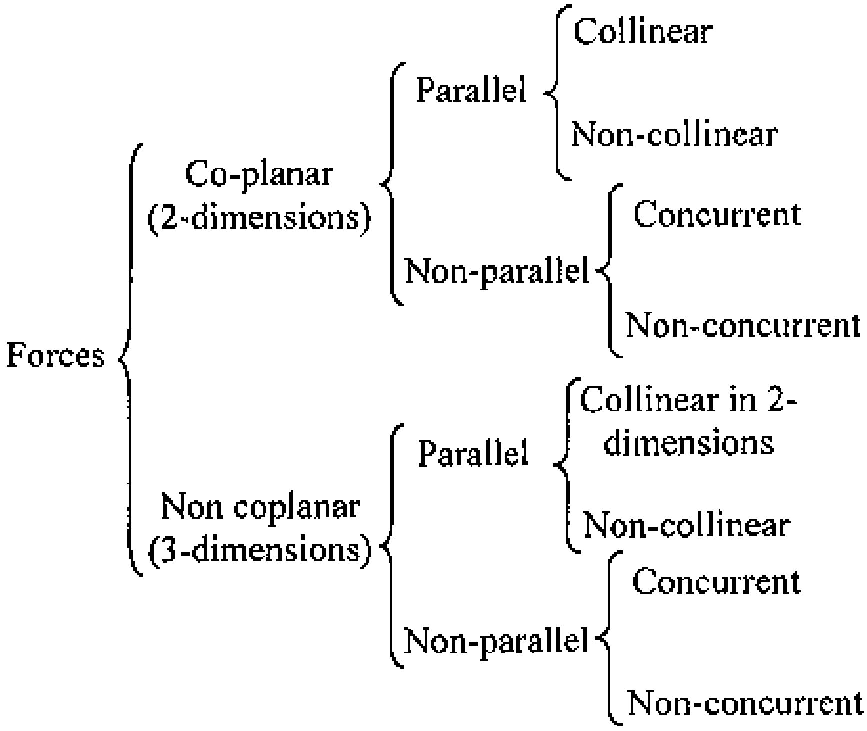


Figure 8: Classification of force system (7 types)

Law of Transmissibility of Forces: It states that the state of rest or motion of a rigid body is unaltered if a force acting on the body is replaced by another force of the same magnitude and direction but acting anywhere on the body in the line of action of the replaced force.

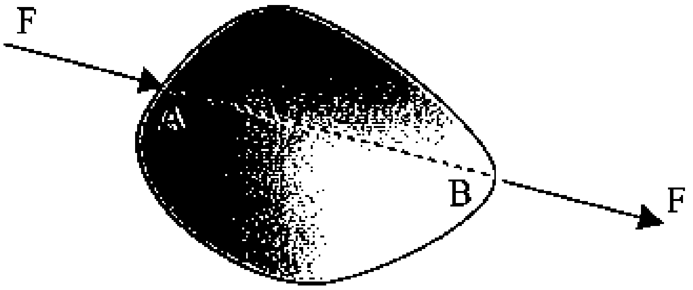


Figure 9: Law of transmissibility of force

In figure 9, F is the force acting on a rigid body at point A. According to the law of transmissibility of forces, if the point of application of this force is shifted to another point B, which is on the line of action of the force F, the state of rest or the steady motion of the body is still unaltered.

The following points are to be carefully noted:

(1) This law is applicable only for rigid bodies, which is the subject matter of engineering mechanics.

(2) In case of subjects like strength of materials in which deformable bodies are concerned, the law of transmissibility will not hold good, since the point of application of the forces completely alter the internal stresses in the deformable bodies.

Moment/Moment of a Force about a Point: A force has not only the tendency to move a body but it has a tendency to rotate the body also. The measure of the rotational effect of a force is called the moment. The moment of a force about a point is defined as the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the point.

$$\text{Moment} = \text{Force} \times \text{perpendicular distance}$$

Units of Moment: N-m, kN-m, in-lb, ft-lb, kip-ft, kip-in & so on.

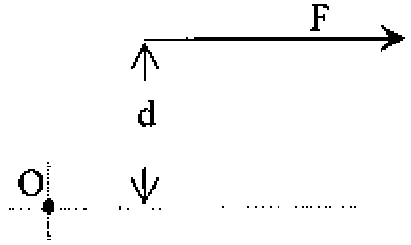


Figure 10(a)

In figure 10(a), moment of force F about point O is

$$M_O = F \times d \text{ (Clockwise)}$$

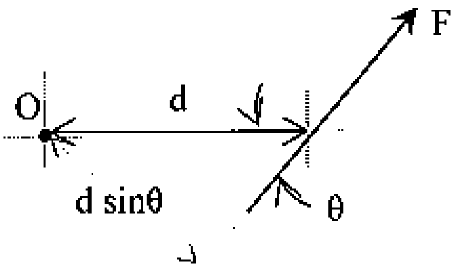


Figure 10(b)

In figure 10(b), moment of force F about point O is

$$M_O = F \times d\sin\theta \text{ (Anti-clockwise)}$$

Types of Forces acting on a Body: While applying equilibrium conditions, it is essential that all forces acting on the body are considered. The various forces acting on the bodies may be grouped as

(1) Applied Forces
(2) Non-applied forces

(1) Applied Forces: These are the forces applied externally to the body. An applied force has got at least one point of contact with the body. If a person stands on the ladder, his weight is an applied force. If a car is pulled by a rope, the force in the rope is an applied force on the car. The usual types of applied forces (also called applied loads) are:

(a) Point Force: If a load is acting on the body over a very small area without much loss of accuracy in the analysis of the body, it may be approximated as force acting at a point. Thus, a concentrated load is idealized 'point force'. The meaning of point force is already discussed (see paragraph idealization in mechanics).

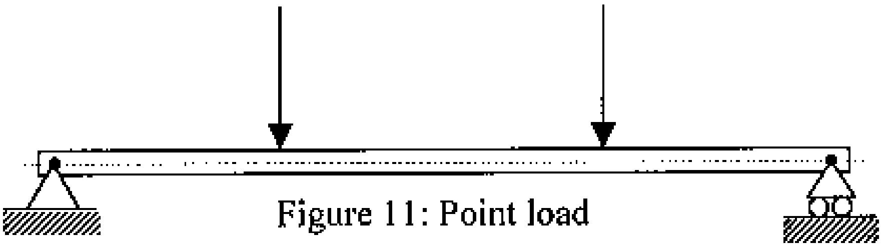


Figure 11: Point load

In actual practice, it is not possible to apply a load at appoint (i.e., a mathematical point); as it must have some contact area. But this area, being so small, as compared to the length of the beam, is neglible.

(b) Linear Force: A force acting along a line of the body is known as the linear force. Figure 12 shows a linear loading on the beam. In this, the ordinate represents the intensity of loading and the abscissa represents the position of the load on the beam. The force dF on an elemental length dx is given by

$$dF = w\,dx$$

Where, w is the intensity of load at a distance x.

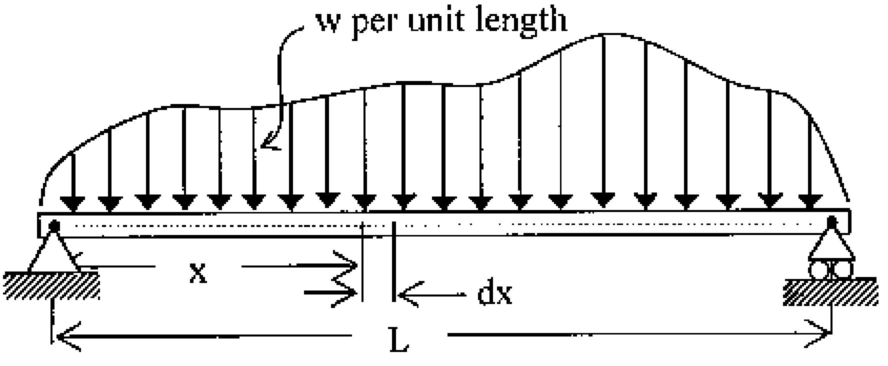


Figure 12: Linear force

The following are the intensity cases of linear forces.

(i) Uniformly distributed load (UDL): A load which is spread over a beam in such a manner that each unit is loaded to the same extent, is known as uniformly distributed load (briefly written as UDL).

(ii) Uniformy varying load (UVL): A load which is spraed over a beam, in such a manner that its extent varies uniformly on each unit length, is known as uniformly varying load (briefly written as UDL).

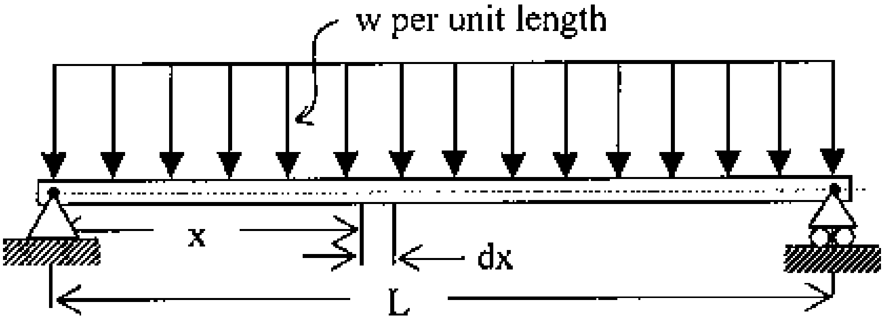


Figure 13: Uniformly distributed load (UDL)

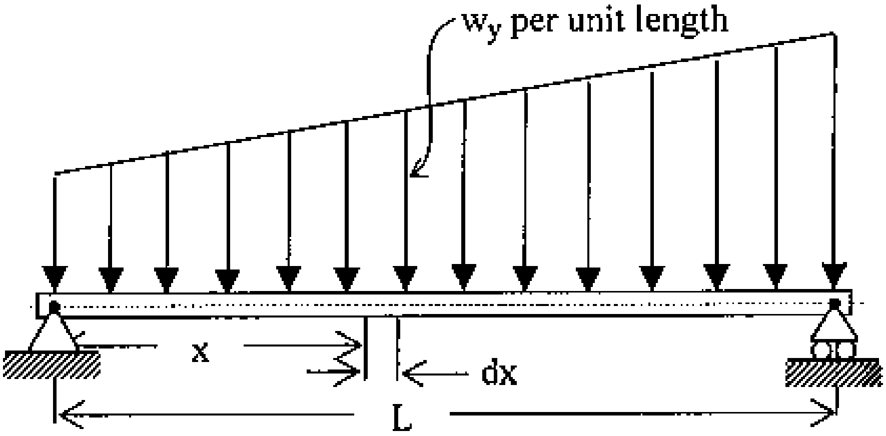


Figure 14(a): Uniformly varying load (UVL)

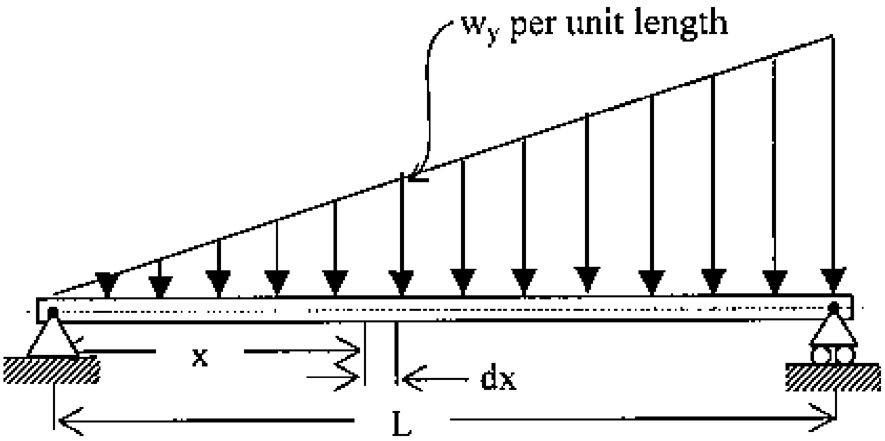


Figure 14(a): Uniformly varying load (UVL) [in order that the beam reactions may be computed, a distributed load must be replaced by its equivalent concentrated resultant load, which always acts through the c.g. (center of gravity) of the load or through the centroid of the loaded area]

(c) Surface Force: A force acting on the surface of a body is termed as the surface force. If p the intensity of load (usually termed as pressure) acting on an elemental area dA, the force acting on the element,

$$dF = p\,dA$$

A common example of such a force is the hydrostatic pressure acting on a body immersed in water.

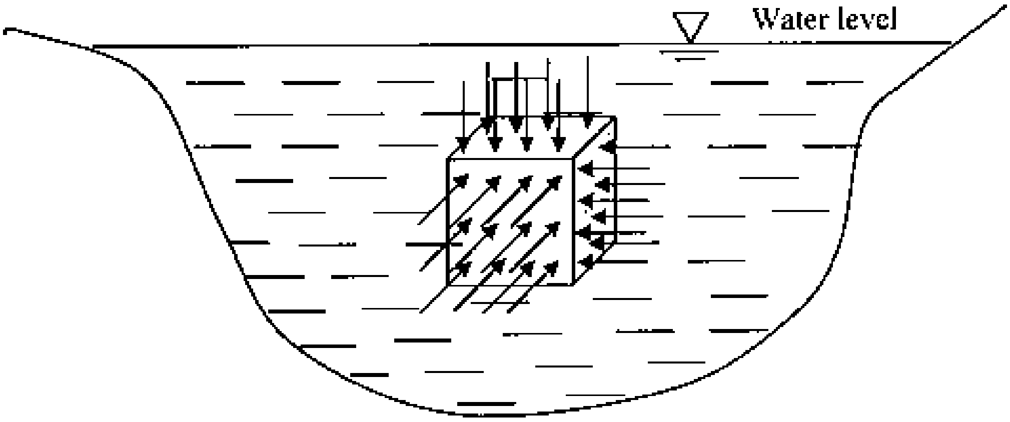
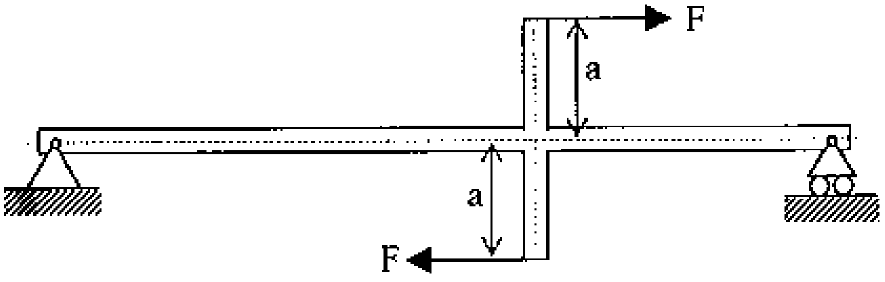


Figure 15: Surface force

(d) External Moment:



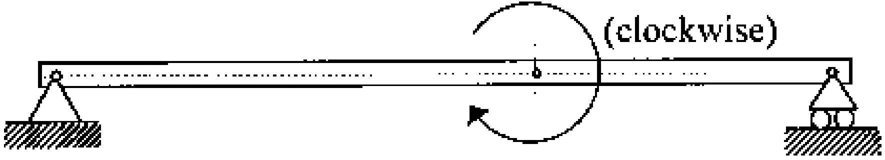


Figure 16: External moment

(2) Non-applied Forces: There are two types of non-applied forces

(a) Self-weight: Every body near the earth is subjected to gravitational acceleration and hence has got a self-weight. The weight of a body W, is given by

$$W = mg$$

Where, m is the mass of the body and g is the gravitational acceleration (near the earth's surface, the gravitational acceleration is 9.81 meter/sec^2 (32.2 feet/sec^2 = 981 cm/sec^2)). Since the self-weight is due to the gravitational acceleration, it always acts towards the center of the earth, i.e. in the vertical downward direction. In the analysis, self-weight is treated as acting through the center of gravity of the body. If the self-weight is very small compared to the applied forces acting on the body, it may be neglected also.

(b) Reactions: Reactions are self-adjusting forces exerted by the other bodies in contact or by the supports. According to Newton's third law of motions, the reactions are equal and opposite to the actions. The reactions always adjust themselves to bring the body to equilibrium. Following paragraph shows different types of contacts between the body and other bodies/supports and also the reactive forces acting on the body under consideration in case of two-dimensional problem.

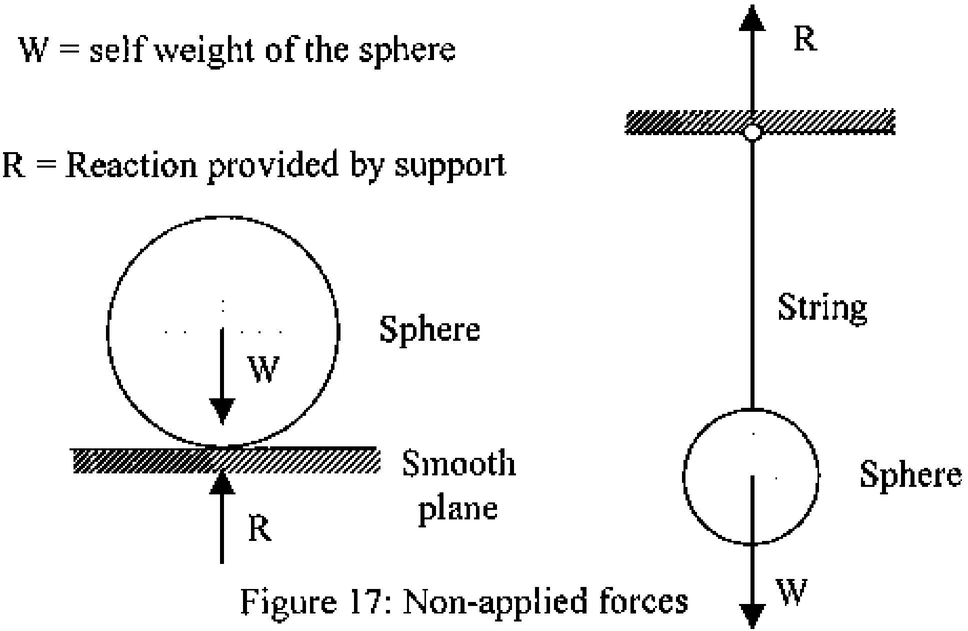

W = self weight of the sphere
R = Reaction provided by support
Sphere
W
Smooth plane
R
R
String
Sphere
W

Figure 17: Non-applied forces (self weight & reaction)

Types of Contacts/Supports:

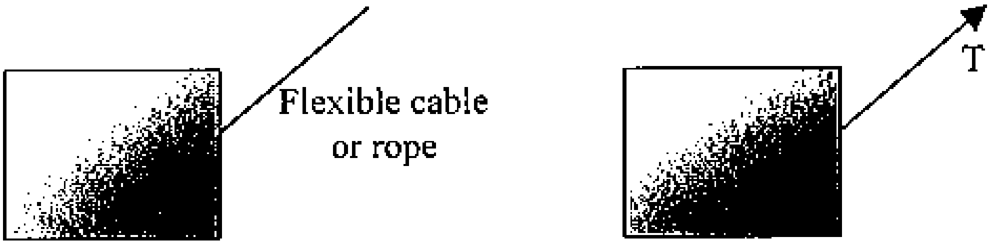


Figure 18: Flexible cable or rope

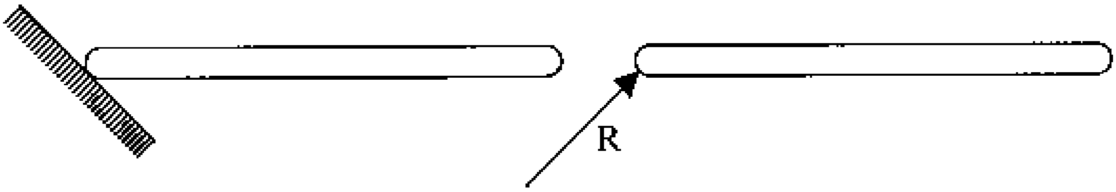


Figure 19: Smooth surface

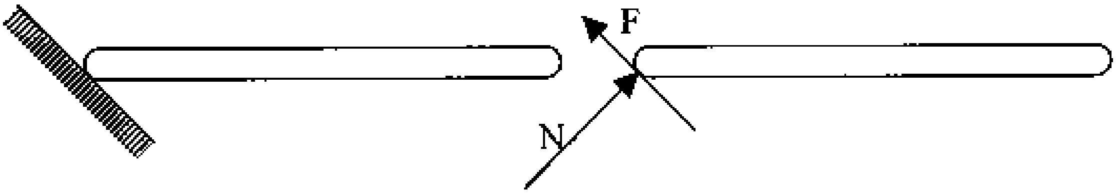


Figure 20: Rough surface

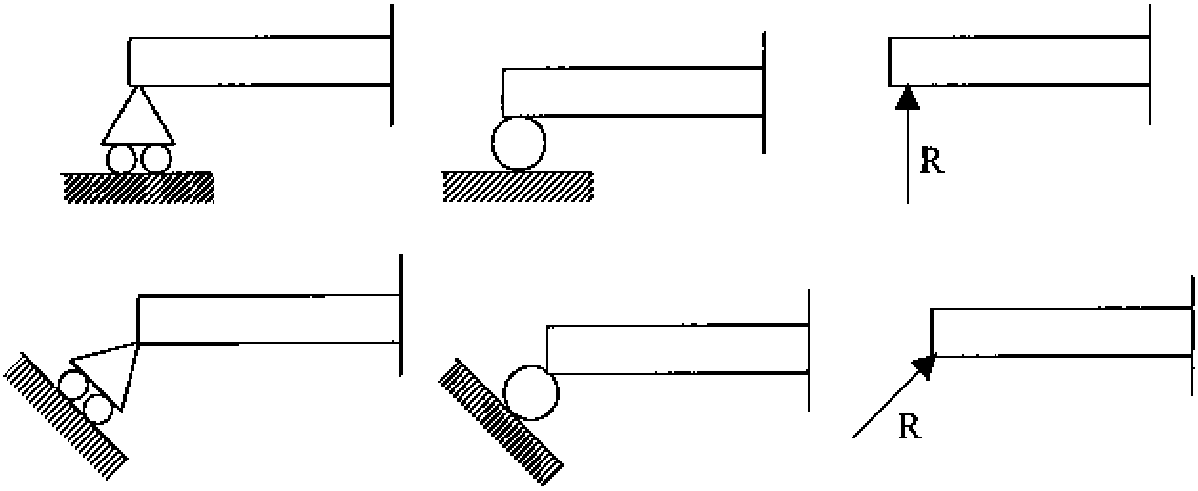


Figure 21: Roller support / Simple support

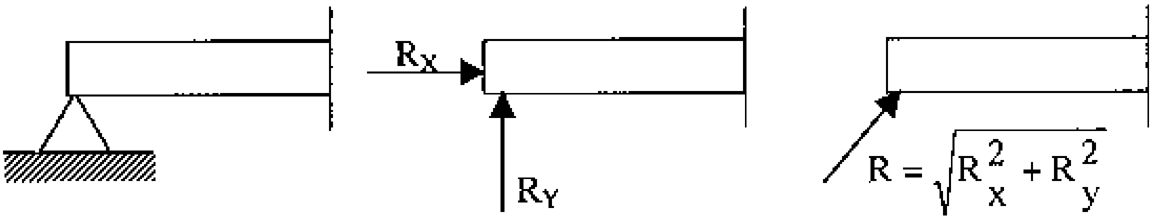


Figure 22: Pinned support / Hinge support

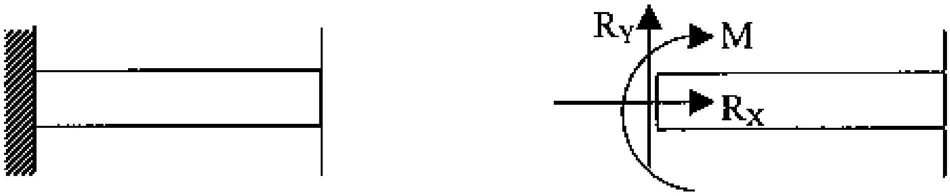


Figure 23: Fixed support / Built-in support

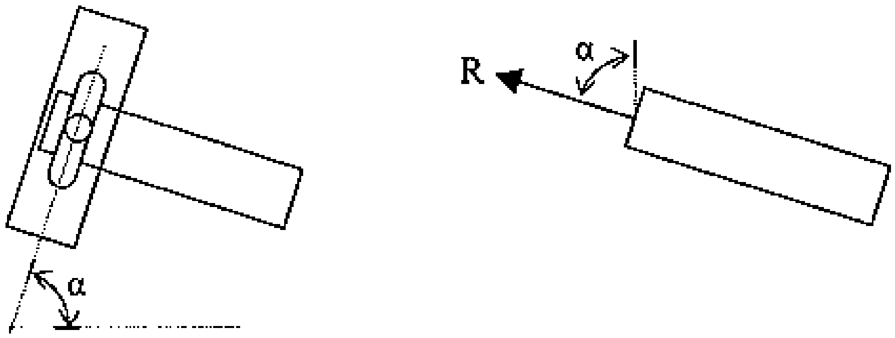


Figure 24: Pin in smooth slot

Table 3: Types of reactive forces in 2-D (two-dimensional) problems

Types of contact/support	See figure no.	Remarks
Flexible cable or rope	Figure 18	Tensile force in cable always away from the body
Smooth surface	Figure 19	Always normal to the surface of contact
Rough surface	Figure 20	Normal to the surface (N) and a frictional force (F) (parallel to the surface) acting so as to oppose the body
Roller support / Simple support	Figure 21	Always normal to support
Pinned support / Hinged support	Figure 22	No moment but reactive force in any direction, which may be conveniently represented by its components in two mutually perpendicular directions
Fixed support / Built-in support	Figure 23	Moment and force in any direction (this force may be conveniently represented by its components in two mutually perpendicular directions)
Pin in smooth slot	Figure 24	One force perpendicular to smooth slot

Free Body Diagrams: A free body diagram shows the action of other bodies on the free body.

Free-body diagrams can be drawn for

- for the entire assembly
- for the component parts

A free-body diagram of an entire structure leads to relationships between the external forces on the structure, whereas free-body diagrams if sub-sections of the structure will lead to relationships between the external and internal forces. In any case, each free-body diagram represents a structural unit that is subjected to all of the considerations of structural analysis. The judicious selection of free–body diagrams, and subsequent analysis of each structural component, is fundamental to the field of structural analysis.

Drawing Free Body Diagrams: Free body diagrams means the sketch of the body separated from other bodies and showing all the forces acting on it including reactions from the other bodies and the self-weight. Drawing the free body diagram is an important step in the analysis of bodies for equilibrium conditions. The steps to be followed in drawing a free body diagram are:

(1) Draw a neat sketch of the body.
(2) Mark all the applied forces.
(3) Mark the self-weight of the body, which may be taken as acting through the centriod of the body. It always acts in the vertical downward direction.
(4) Mark reactive forces, which may be from the supports or from the other bodies in contact with the body under consideration while marking these forces, follow the guidelines given in table 3.
(5) Apply equations of statics and solve for unknowns.

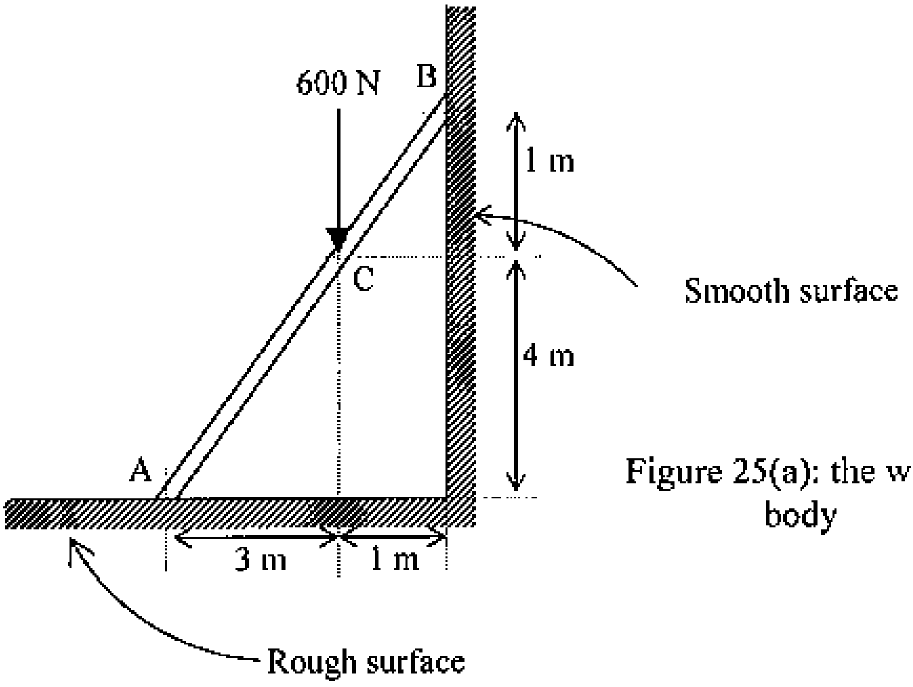


Figure 25(a): the whole body

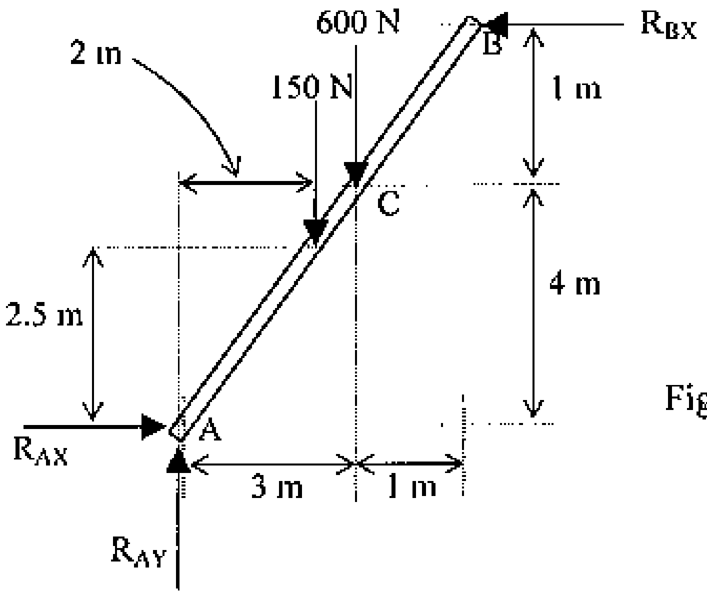
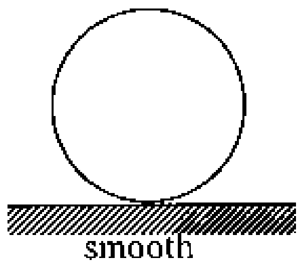
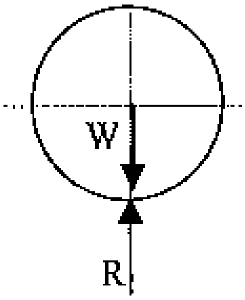


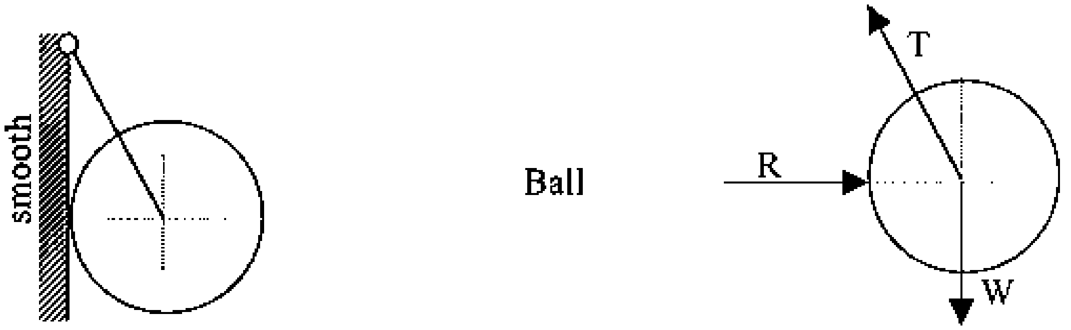
Figure 25(b): free body diagram of ladder

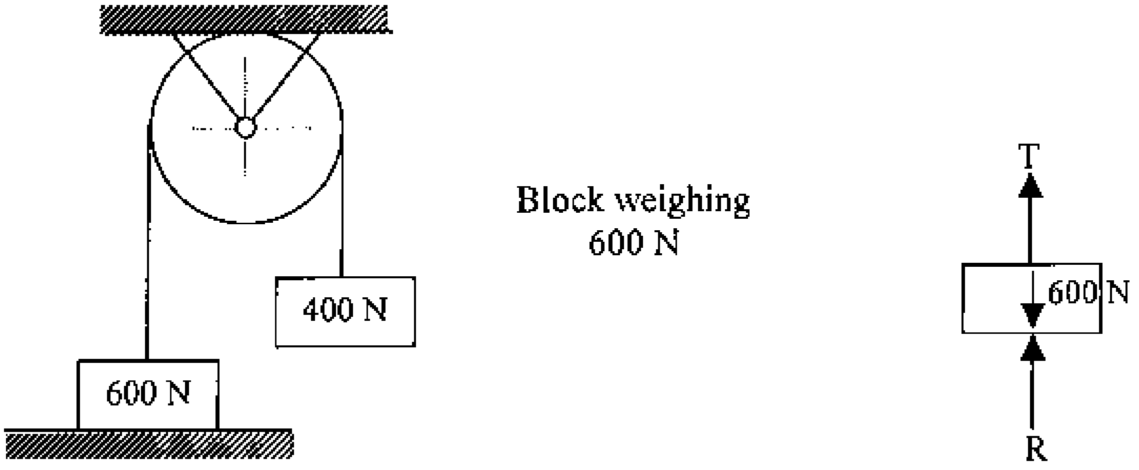
For example, consider the ladder shown in figure 25(a), which weighs 150 N. On this, a person weighing 600 N is standing at standing at point C. The wall is smooth and the floor is rough. To draw the fee body diagram of the ladder, consider the following steps:

(1) Draw the ladder.
(2) Draw the applied load of 700 N at C, which acts vertically downward since it is the weight.
(3) Self-weight of 150 N is drawn through the centroid of the ladder.
(4) The reaction of wall R_{BX} is the horizontal force, since the surface of contact is smooth and vertical. The reaction at floor may be represented by the normal reaction to floor R_{AY} and the horizontal frictional force R_{AX}.

Table 4: Free body diagrams for typical cases

Reacting bodies	Free body diagram required for	Free body diagram
 smooth	Ball	 W R





Approaches for Analysis: There are two different approaches used for the analysis of engineering mechanics problems, namely *Classical Approach* (graphical method, algebraic method & matrix method) and *Vector Approach*.

In classical approach, the equations are written down only for magnitude, and direction is noted down separately for each equation; in vector approach, the direction is also incorporated in the equation using vector algebra. The classical approach helps in developing a pictorial view of the analysis, and hence there will be excellent development of the analytical mind. This approach is very well suited for two-dimensional problems. However, for three-dimensional problems, it is difficult to imagine pictures and keep track of directions. Hence, they can be conveniently handled by vector approach. For a beginner, the approach should be physical rather than mathematical.

Laws of Statics/ Equations of Static Equilibrium: A structural is said to be in equilibrium* when it is initially at rest and remains at rest as it is acted upon by a set of forces. When a structure satisfies this test for equilibrium, any constituent part of the structure will also be found to be in equilibrium.
[*Equilibrium is a special type of motion in which there is no acceleration for any particles of the system. A body moving with constant velocity (i.e., no acceleration) is an example of equilibrium]
For the condition of rest to be satisfied, there must be a balance in the forces tendencies that would act to disturb the structure in any way. The balance can be

expressed mathematically for 2-D (two-dimensional) & 3-D (three-dimensional) structure as below:

For the whole or any part of a uniplanar / two-dimensional structure, two conditions must be satisfied for static equilibrium.
(a) The algebraic sum of all forces resolved in any direction equals zero.
(b) The algebraic sum of all the moments of all forces about any point in the same plane equals zero.

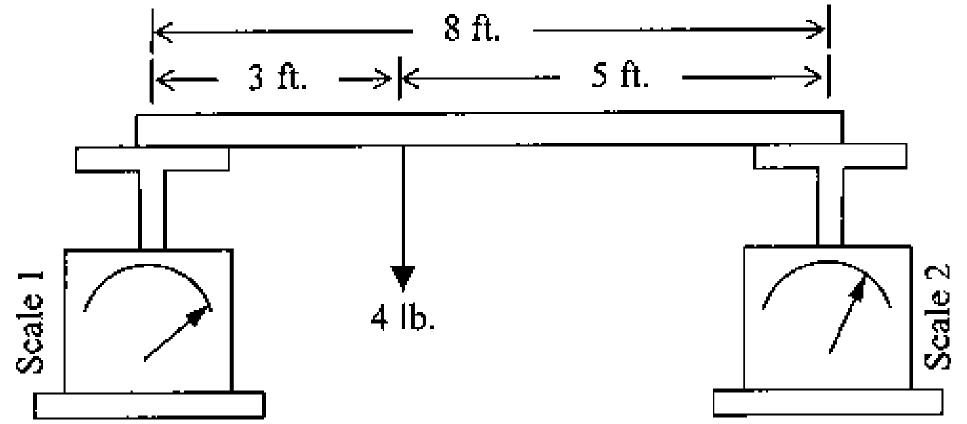
Mathematically, they are expressed as
$\sum F_X = 0$ ($\sum$ forces in horizontal or X direction = 0)
$\sum F_Y = 0$ ($\sum$ forces in vertical or Y direction = 0)
$\sum M_Z = 0$ ($\sum$ moment about any point = 0)

For a space / three-dimensional structure, the conditions which must be satisfied for static equilibrium are
(a) The algebraic sum of all external forces resolved in any direction equals zero.
(b) The algebraic sum of all the moments of all external forces about any axis equals zero.

Mathematically, they are expressed as
$\sum F_X = 0$ ($\sum$ forces in X direction = 0)
$\sum F_Y = 0$ ($\sum$ forces in Y direction = 0)
$\sum F_Z = 0$ ($\sum$ forces in Z direction = 0)
$\sum M_X = 0$ ($\sum$ moment about X axis = 0)
$\sum M_Y = 0$ ($\sum$ moment about Y axis = 0)
$\sum M_Z = 0$ ($\sum$ moment about Z axis = 0)

These equations are referred to as the equations of static equilibrium/laws of statics. When these equations are satisfied, the body is in balance, at rest, and thus in equilibrium.

Experiment on Laws of Equilibrium: See page no 81 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St Paul's House Warwick Lane, London EC4.



This scale shows a
reading of 3.5 lb.

This scale shows a
reading of 2.5 lb.

Figure 26: Demonstration of beam support reactions

Demonstration of Beam Support Reactions: Place the ends of a small beam on the two scales (see figure 26). Apply a 4 lb. load at a distance 3 ft from the left end and observe that the scales readin60
gs. The scales on the left end and right end show a value of 3.5 lb. & 2.5 lb. respectively. Checking these results by the laws of statics, we have:

$R_1 \times 8 = 4 \times 5 = 20$; $R_1 = 3.5$ lb.
$R_2 \times 8 = 4 \times 3 = 12$; $R_2 = 2.5$ lb.
$R_1 + R_2 = 3.5 + 2.5 = 4$ lb. (= applied load)

Procedure in the Solution of Mechanics Problems: Successful and efficient solution of any engineering problem calls for a well-organized and logical method of attack, involving a number of steps, each of which must first be well understood and then carefully executed. Among these steps, the following five include in a general way the entire process of solving any problem:

1. Analyze carefully the given data and ascertain the known quantities and the unknown quantities to be determined.
2. Recognize all the acting forces, known & unknown.
3. Decide on a suitable type of solution to use to determine the unknown quantities.
4. Formulate the steps to be taken to complete this solution.
5. Execute these steps, using available methods of checking the results.

The necessity for checking intermediate as well as final results as the solution progresses cannot be overemphasized, and yet it is most difficult to impress this fact on students, especially during the early part of their training; too many insist on dashing on to some answer, often finding it to be wrong and then, on a recheck, discovering some foolish or careless mistake which a second glance at the proper time would have quickly revealed.

Standards for Workmanship in Problem Solution: Because of the great responsibility that attacks to the practice of engineering, high standards of workmanship are demanded by the profession. These standards call for clear and neat figures and letters, and for crowded and logical arrangement of all computations and diagrams. Squared paper, 8.5 inch × 11 inch and a straightedge for drawing diagrams are recommended. All diagrams must be complete; that is they must show all forces, dimensions, and other items that are parts of the problem.

In order that computations may readily be checked, as they must be, all work except the simplest additions, subtractions, multiplications, and divisions must be shown. Of course, if the slide rule is used, no multiplications or divisions need be shown, but the processes must be indicated. One of the best ways of checking data and computations is to glance over them as soon as they are completed, to see that no mistake has been made and that the result obtained so far is reasonable. Often an absurd answer or a misplaced decimal point is thus quickly detected.

Accuracy of Calculations: There is no point to in noting down the calculations and final answers beyond four digits. However, higher accuracy may be maintained for intermediate steps, since some of them may affect final values adversely.

Useful Data: The following data summarizes the previous memory and formulae, the knowledge of which is very essential at this stage.

Algebra:

(1) $a^0 = 1,\ x^0 = 1$

(2) $x^m \times x^n = x^{m+n}$

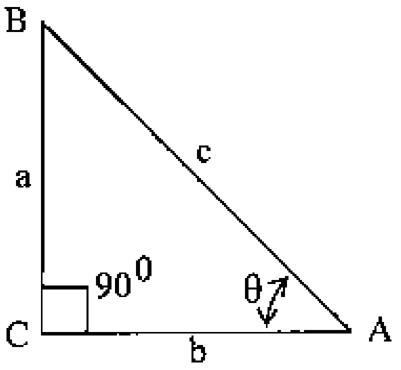
(3) $\dfrac{x^m}{x^n} = x^{m-n}$

(4) If the quadratic equation $ax^2 + bx + c = 0$, the roots of the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where, a is the co-efficient of x^2, b is the co-efficient of x & c is the constant term.

Trigonometry: In a right-angled triangle ABC



AB = c = hypotenuse, AC = b = base, BC = a = perpendicular

$(\text{perpendicular})^2 + (\text{base})^2 = (\text{hypotenuse})^2$

$=> a^2 + b^2 = c^2$

$=> c = \sqrt{a^2 + b^2}$; this is called Pythagoras theorem

$$\sin\theta = \frac{a}{c},\ \cos\theta = \frac{b}{c},\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(a/c)}{(b/c)} = \frac{a}{b}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a},\ \sec\theta = \frac{1}{\cos\theta} = \frac{c}{b},\ \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{(b/c)}{(a/c)} = \frac{b}{a}$$

Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = c^2 + a^2 - 2ca\cos B$

$c^2 = b^2 + a^2 - 2ba\cos C$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Functions of angles:

$(\sin\theta)^2 + (\cos\theta)^2 = 1$

$=> \sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta = 1 + \tan^2\theta$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$2\cos^2\theta = 1 + \cos 2\theta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Following tables shows values of some trigonmetrical functions for some typical angles.

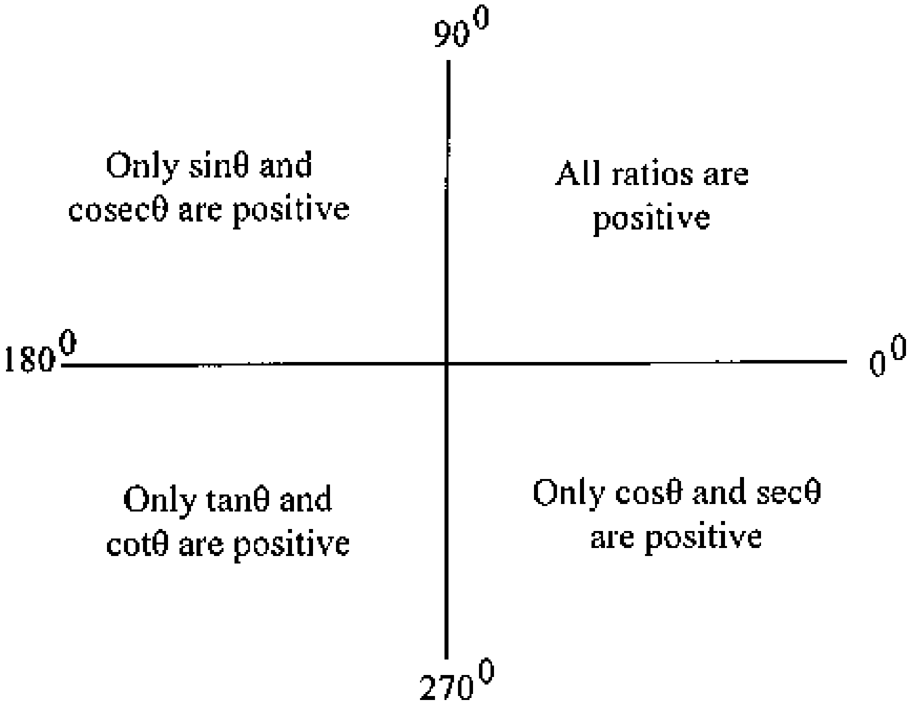
Angle	0^0	30^0	45^0	60^0	90^0
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	---

In the first quadrant $\left(0^0 - 90^0\right)$, all the trigonmetrical ratios are positive.

In the second quadrant $\left(90^0 - 180^0\right)$, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive.

In the third quadrant $\left(180^0 - 270^0\right)$, only $\tan\theta$ and $\cot\theta$ are positive.

In the fourth quadrant $\left(270^0 - 360^0\right)$, only $\cos\theta$ and $\sec\theta$ are positive.

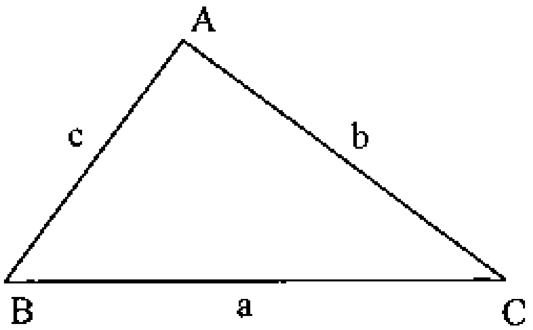

90^0
Only sinθ and cosecθ are positive
All ratios are positive
180^0
0^0
Only tanθ and cotθ are positive
Only cosθ and secθ are positive
270^0

In a triangle ABC, BC = a, CA = b, AB = c and s = half perimeter, Δ = area of the triangle.

$$s = \frac{a+b+c}{2}$$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$; Heron's formula – a formula connecting the area of a triangle with its sides.

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$$



Useful Conversion Factors:

1 foot = 12 inch
1 yard = 3 feet
1 meter = 3.28 feet
1 inch = 2.54 centimeter = 25 mm.
1 mile = 1760 yard = 5280 feet
1 foot = 30.48 centimeter
1 mile = 1.609 kilometer

1 kilometer = 0.621 mile

1 acre = 43560 square feet
1 square foot = 144 square inch
1 square meter = 10.76 square feet
1 square mile = 640 acres

1 cubic foot = 1728 cubic inch
1 gallon = 7.481 cubic feet
1 cubic feet = 28320 cubic centimeter
1 gallon = 231 cubic inch
1 cubic meter = 3531 cubic feet

180 degree = π radian = 60 grade
1 degree = (π/180) radian
1 radian = (180/π) degree

1 day = 24 hours
1 hour = 60 minutes
1 minute = 60 seconds

1 knot = 1.152 mile per hour
1 mile per hour = 88 feet per minute
1 mile per hour = 1.467 feet per second

Table 5: Basic Greek Symbols:

Name	Symbol
Alpha	α
Beta	β
Chi	χ
Delta	δ
Epsilon	ε
Eta	η
Gamma	γ
Iota	ι
Kappa	κ
Lambda	λ
Mu	μ
Nu	ν
Omega	ω
Phi	φ
Pi	π

Psi	ψ
Rho	ρ
Sigma	σ
Tau	τ
Theta	θ
Xi	ξ
Zeta	ς

$\frac{223}{71} < \pi < \frac{22}{7}$; $3.1415926 < \pi < 3.1415797$

$\pi = 3.14159\ 26535\ 89793\ 23864\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510$

Selected Rules & Suggestion for SI Usage:

(1) Care must be taken to use the correct case for symbols, units and prefixes (e.g., m for meter or milli, M for mega)

(2) For numbers having five or more digits, the digits should be placed in groups or three separated by a space, counting both to the left and to the right of the decimal point (e.g., 61 354.982 03). The space is not required for four-digit members. Spaces are used instead of commas to avoid confusion – many countries use the comma as the decimal marker.

(3) In computed units formed by multiplication, use the product dot (e.g., N.m).

(4) Division may be indicated by solidus (m/s), or a negative exponent with a product dot ($\mathrm{m.s^{-1}}$)

(5) Avoid the use of prefixes in the denominator (e.g., km/s preferred over m/ms). The exception to this rule is the prefix k in the base unit kg (kilogram).

Chapter Two
Resultants & Components

Resultant & Components: If a number of forces P_1, P_2, P_3 --- --- etc., are acting simultaneously on a particle, then a single force, which will produce the same effect as that of all the given forces, is known as **a resultant force.** The forces P_1, P_2, P_3 --- --- etc. are called **component forces.**

As discussed previously, a force is that it can push or pull a rigid body. Now suppose that the rigid body is pushed or pulled by more than one force at a time. The effect of the forces acting together should be the same as the sum of the effects of each on acting separately. The process of finding the sum of the effects of all the separate forces is called **composition or addition.** The answer obtained when forces are added together is called the **resultant force.** The single resultant force will do everything to the rigid body that each of the separate forces acting individually and simultaneously will do. In a great many problems much worked can be saved by adding a lot of forces together into a single resultant, and then dealing with the single force rather than with a lot of separate forces.

The process of adding forces together must be done in accordance with the parallelogram law of addition, because forces are vector quantities.

The process of **composition (addition)** of forces may be reversed into a process called **resolution** (breaking apart) (i.e. the process of finding components of a force called resolution). Each of the forces into which a given force is broken is called a **component force.** In contrast to situation where a set of forces could be combined into only one possible resultant, a given force can be broken up into an infinite number of possible components.

Rectangular Components /Orthogonal Components: Of all possible combination of components, only one is much importance in statics. This combination is arranged in such a way that there are only two components in the combination. Further than that the two components are chosen so that they are at right angles to each other. For this reason, they are often called **rectangular/orthogonal components.**

Parallelogram Law of Forces: This law was formulated based on experimental results. Though Stevinus employed it in 1586, the credit of presenting it as a law goes to Varignon and Newton (1687). The parallelogram law of forces enables so to replace the two forces acting at a point by a single force called the resultant force acting at that point without altering any effect on the body.

It states, "If two forces acting simultaneously on a particle be represented, in magnitude and direction, by the two adjacent sides of a parallelogram, their resultant may be represented, in magnitude and direction, by the diagonal of the parallelogram passing through the point of their intersection."

F_1 = One component force

F_2 = Another component force

R = Resultant of two components forces F_1 and F_2

α = Angle between two components forces F_1 and F_2

θ_1 = Angle between resultant force R and component force F_1

θ_2 = Angle between resultant force R and component force F_2

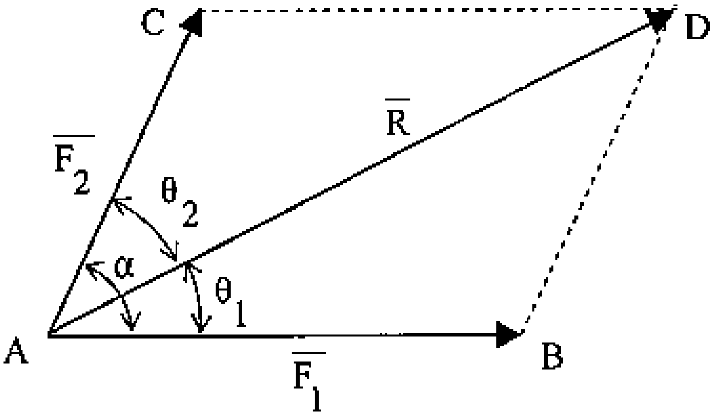


Figure 1: Parallelogram law

$$R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\alpha$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

$$\tan\theta_1 = \frac{F_2\sin\alpha}{F_1 + F_2\cos\alpha} \ \& \ \tan\theta_2 = \frac{F_1\sin\alpha}{F_2 + F_1\cos\alpha}$$

If more than two forces are acting on a body, two forces at a time can be combined by the parallelogram law of forces and finally the resultant of all the forces acting on the body can be obtained.

Example (1): Find out the magnitude and resultant of two forces acting simultaneously on a particle at point A.

Solution (1): F_1 = One component force = AB = 500 N

F_2 = Another component force = AC = 400 N

α = Angle between two components forces F_1 and $F_2 = 60^0$

R = Resultant of two components forces $F_1 = 500$ N and $F_2 = 400$ N

θ_1 = Angle between resultant force R and component force F_1

θ_2 = Angle between resultant force R and component force F_2

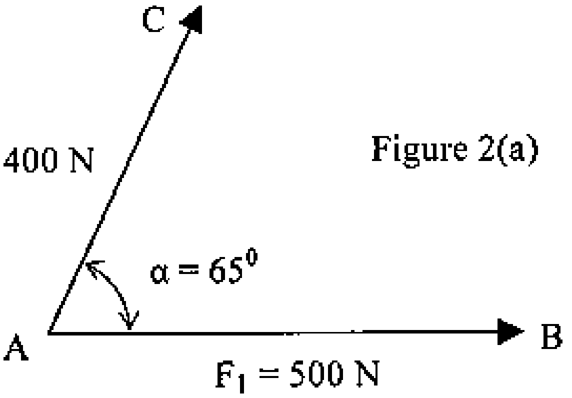


Figure 2(a)

$$R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\alpha$$

$$\Rightarrow R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

$$\Rightarrow R = \sqrt{(500)^2 + (400)^2 + 2(500)(400)(\cos 65^0)}$$

$$\Rightarrow R = 760.9515 \text{ N}$$

$$\tan\theta_1 = \frac{F_2\sin\alpha}{F_1 + F_2\cos\alpha} = \frac{400\sin 65^0}{500 + 400\cos 65^0} = 0.54184975$$

$$\theta_1 = \tan^{-1}(0.54184975) = 28.45103^0$$

$$\tan\theta_2 = \frac{F_1\sin\alpha}{F_2 + F_1\cos\alpha} = \frac{500\sin 65^0}{400 + 500\cos 65^0} = 0.741284352$$

$$\theta_2 = \tan^{-1}(0.741284352) = 36.54896^0$$

Check: $\theta_1 + \theta_2 = 28.45103^0 + 36.54896^0 = 65^0 = \alpha$

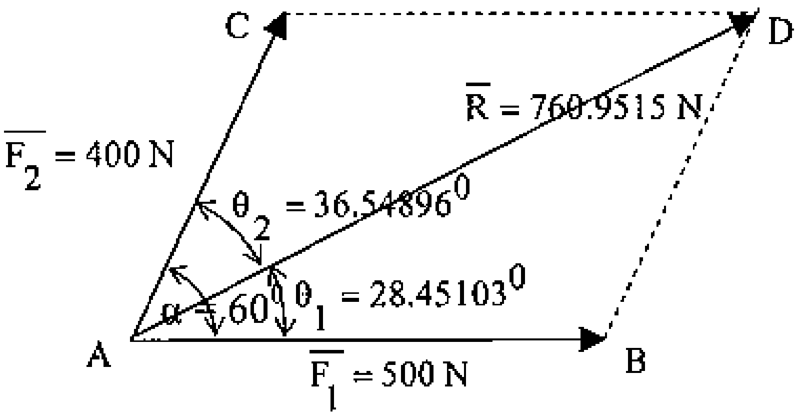


Figure 2(b): Solution of example (1)

Rectangular Components /Orthogonal Components: For $\alpha = 90^0$, we get the special case of components which are perpendicular to each other.

$$R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\alpha$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

For $\alpha = 90^0$, $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos 90^0} = \sqrt{F_1^2 + F_2^2}$

$$R^2 = F_1^2 + F_2^2$$

$F_1 = R\cos\theta$ and $F_2 = R\sin\theta$, $R^2 = (F_1\cos\theta)^2 + (F_2\sin\theta)^2 = F_1^2 + F_2^2$

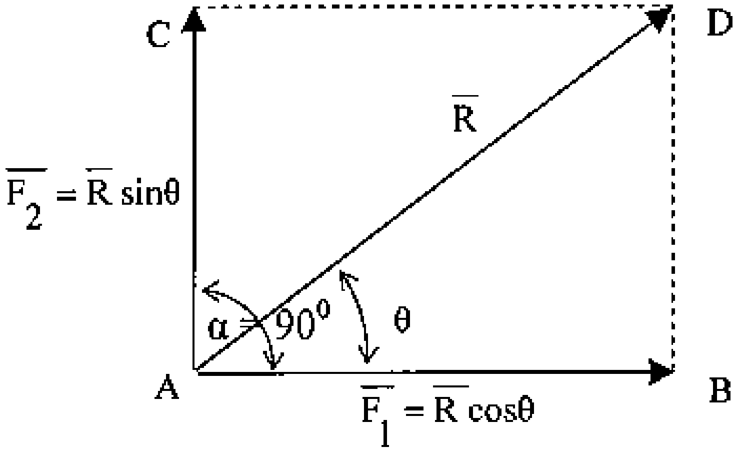

C
D
$\overline{R}$
$\overline{F_2} = \overline{R}\sin\theta$
α
90^0
θ
A
B
$\overline{F_1} = \overline{R}\cos\theta$

Figure 3: Rectangular or orthogonal components

Components of a resultant that are at right ankles to each other are called rectangular or orthogonal components; when they are in the direction of reference axes X and Y, they are usually designated by F_x and F_y.

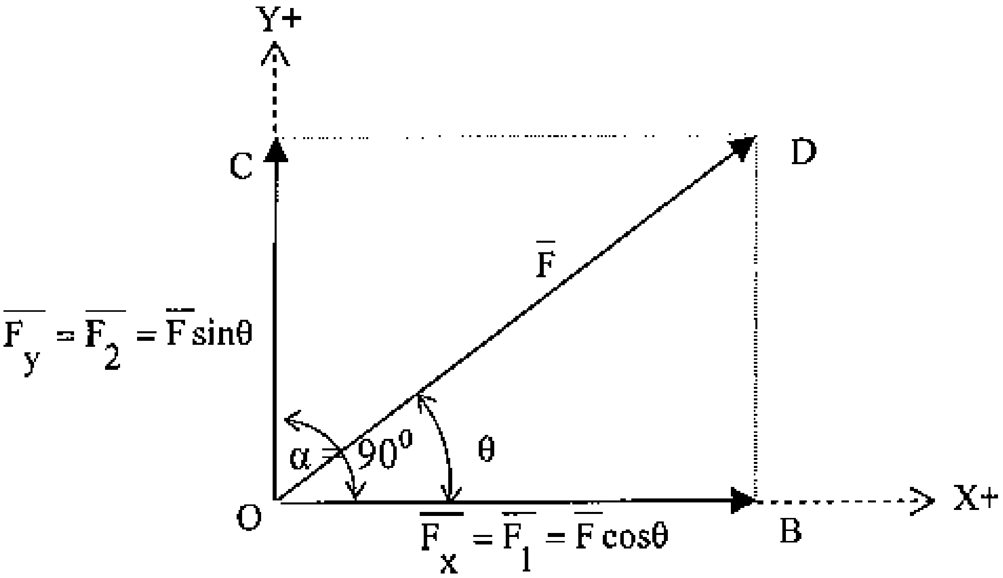

Y+
C
D
$\overline{F}$
$\overline{F_y} = \overline{F_2} = \overline{F}\sin\theta$
α
90°
θ
O
X+
B
$\overline{F_x} = \overline{F_1} = \overline{F}\cos\theta$

Figure 4: Rectangular or orthogonal components

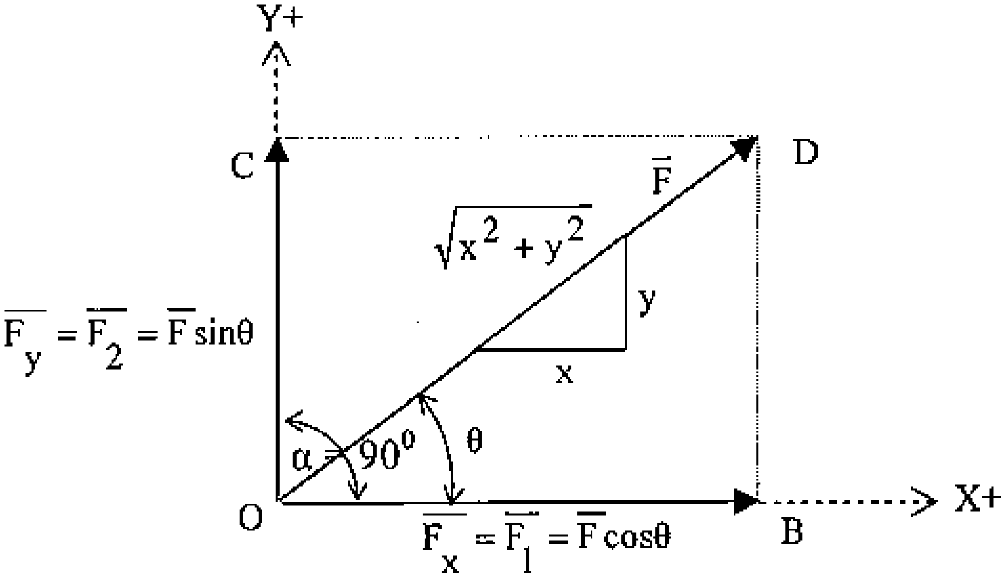

Y+
C
D
$\vec{F}$
$\sqrt{x^2+y^2}$
y
x
$\overline{F_y} = \overline{F_2} = \overline{F}\sin\theta$
α
90°
θ
O
X+
B
$\overline{F_x} = \overline{F_1} = \overline{F}\cos\theta$

Figure 5: Rectangular or orthogonal components

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}};\ \cos\theta = \frac{x}{\sqrt{x^2+y^2}};\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(y/\sqrt{x^2+y^2})}{(x/\sqrt{x^2+y^2})} = \frac{y}{x}$$

$$\theta = \sin^{-1}\frac{y}{\sqrt{x^2+y^2}} = \cos^{-1}\frac{x}{\sqrt{x^2+y^2}} = \tan^{-1}\frac{y}{x}$$

$$F_x = F\cos\theta = F \times \frac{x}{\sqrt{x^2+y^2}}$$

$$F_y = F\sin\theta = F \times \frac{y}{\sqrt{x^2+y^2}}$$

Example (2): Compute rectangular (horizontal & vertical) components of the force 100 N shown below.

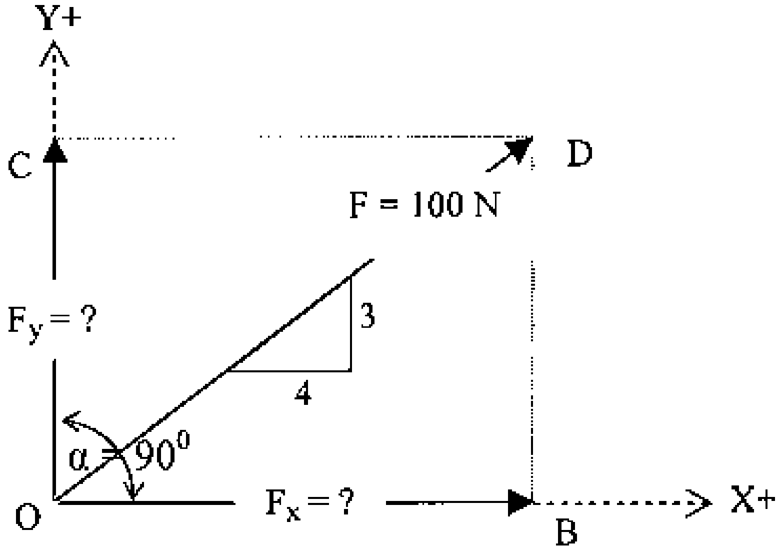
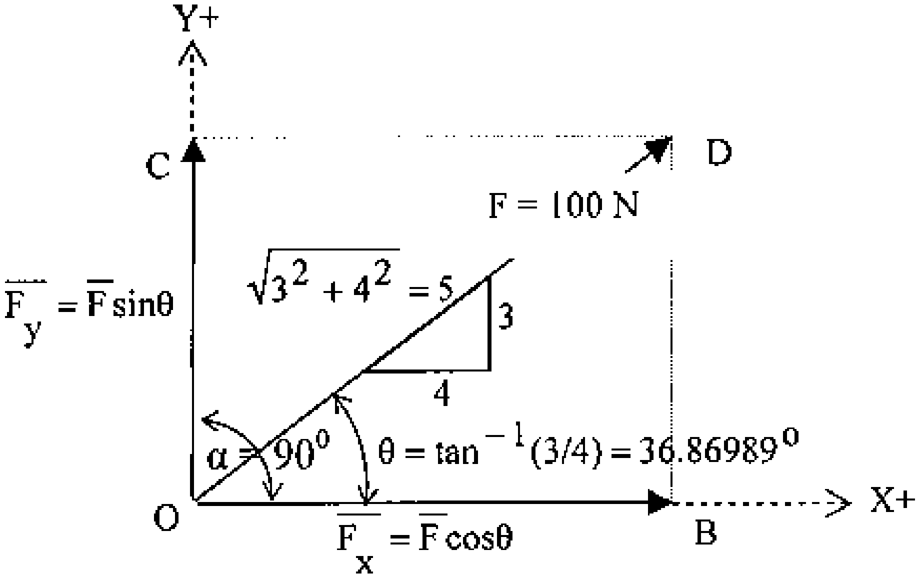


Figure 6(a)



Y+
C
D
F = 100 N
$\overline{F_y} = \overline{F}\sin\theta$
$\sqrt{3^2 + 4^2} = 5$
3
4
α
90^0
$\theta = \tan^{-1}(3/4) = 36.86989^0$
O
$\overline{F_x} = \overline{F}\cos\theta$
X+
B

Figure 6(b)

Solution (2):

Method 1:

$$F_x = F \times \frac{x}{\sqrt{x^2 + y^2}} = 100 \times \frac{4}{\sqrt{4^2 + 3^2}} = 100 \times \frac{4}{5} = 80 \text{ N } (\rightarrow)$$

$$F_y = F \times \frac{y}{\sqrt{x^2 + y^2}} = 100 \times \frac{3}{\sqrt{4^2 + 3^2}} = 100 \times \frac{3}{5} = 60 \text{ N } (\uparrow)$$

Method 2:

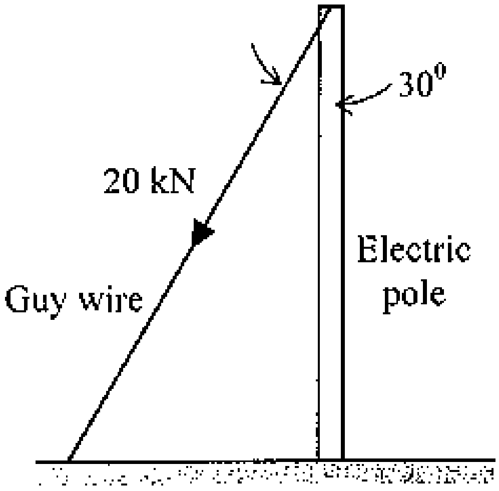
$$\tan\theta = \frac{y}{x} = \frac{3}{4}$$

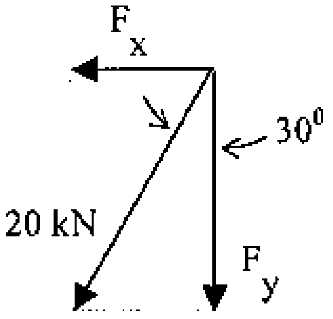
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}(0.75) = 36.86989^{0}$$

$$F_x = F\cos\theta = 100\times\cos(36.86989^{0}) = 80\text{ N } (\rightarrow)$$

$$F_y = F\sin\theta = 100\times\sin(36.86989^{0}) = 60\text{ N } (\uparrow)$$

Example (3): The guy wire of an electric pole makes an angle 30^0 to the pole and is subjected to 20 kN force. Find the vertical and horizontal components of the force.



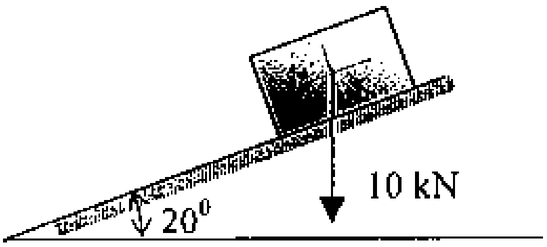


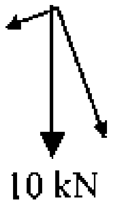
Example (3)

Solution (3): $F_x = 20\times\sin 30^0 = 10$ kN.

& $F_x = 20\times\cos 30^0 = 17.32050808$ kN.

Example (4): A block weighing 10 kN is resting on an inclined plane. Determine its components normal to, and parallel to the inclined plane. The plane makes an angle of 20^0 with the horizontal.





Example (4)

Solution (4): Component of weight normal to the plane = 10 × cos 20^0 = 9.396926208 kN & component of weight parallel to the plane = 10 × sin 20^0 = 3.420201433 kN.

Experimental Verification of Parallelogram Law of Forces: See page no 8 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St. Paul's House Warwick Lane, London EC4.

Equation of Statics / Laws of Equilibrium: See previous chapter.

Lami's Theorem & its limitation: If a body is in equilibrium under the action of three concurrent coplanar forces, each force is proportional to the sine of the angle between the other two forces.

Thus, for the system of forces shown in figure 9.

$$\frac{P_1}{\sin\alpha} = \frac{P_2}{\sin\beta} = \frac{P_3}{\sin\gamma}$$

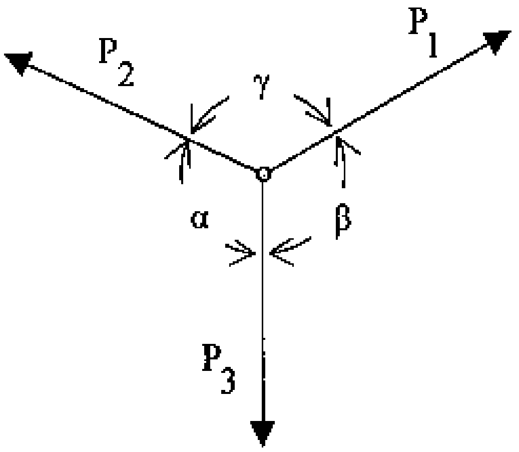


Figure 9: Lami's theorem

If a body is in equilibrium under the action of a number of forces, it may be analysed using the equations of statics ($\sum F_x = 0$ and $\sum F_y = 0$). Lami's theory is applicable only if the body is in equilibrium under the action of three coplanar concurrent forces.

Note: If the magnitude of reaction comes out to be negative, then assumed direction of reaction is wrong. It is acting exactly in the opposite to the assumed direction. However, the magnitude will be the same. Hence, no further analysis is required. This advantage is not there in using Lami's equation. Hence, it is advisable for beginners to use equations of equilibrium, instead of Lami's theorem even if the body is in equilibrium under the action of only three forces.

Note: If two forces pass through one point, the third force must pass through that point to make it equilibrium (i.e. those three forces must be concurrent).

Problems

Problem (1): A boat is held against the current of a stream. The force in the stream is 100 lb to the right. Determine the tension in the two tie lines AB and BC.
Answer (1): Tension in AB = 56.6947 lb & tension in BC = 75.5929 lb.

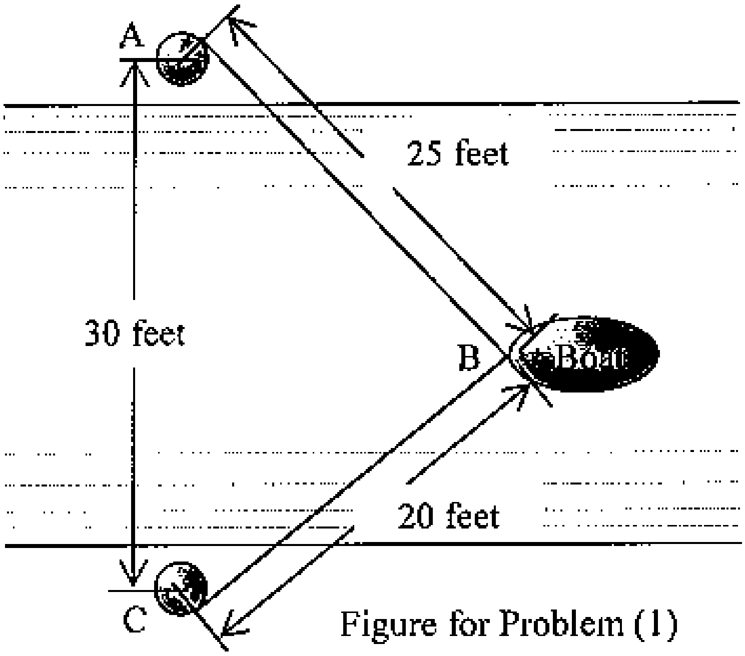
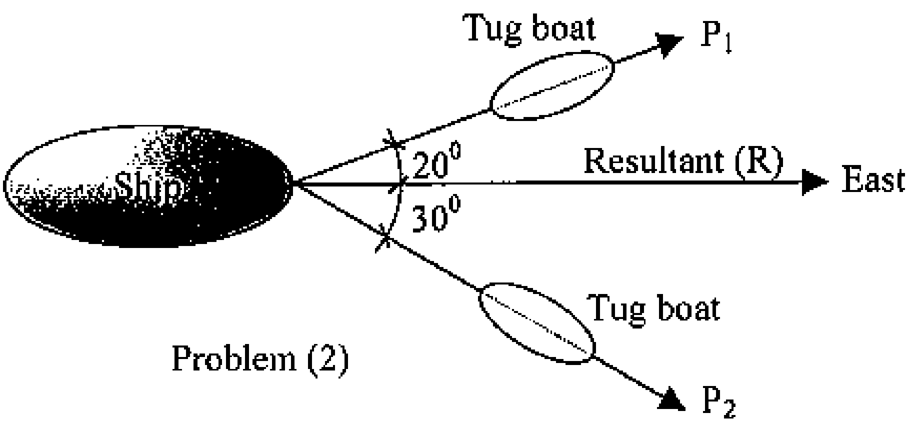


Figure for Problem (1)

Problem (2): A ship is to be towed to shore using two tug boats. If one tug boat is applying a horizontal force at 20^0 and another at 30^0 to the east, determine the ratio of forces applied in two tug boats and the net force acting in the direction of east.



Problem (2)

Solution (2): Summing the forces in a direction perpendicular to the east (↑+ve)

$$\Rightarrow R\cos 90^0 = P_1 \sin 20^0 - P_2 \sin 30^0$$

$$\Rightarrow 0 = P_1 \sin 20^0 - P_2 \sin 30^0$$

$$\Rightarrow \frac{P_1}{P_2} = \frac{\sin 30^0}{\sin 20^0} = 1.4619022$$

Summing the forces in a east direction (→+ve)

$$\Rightarrow R\cos 0^0 = P_1 \cos 20^0 + P_2 \cos 30^0$$

$\Rightarrow R = P_1 \cos 20^0 + P_2 \cos 30^0 = 1.532088886\ P_1$ or $2.239764114\ P_2$ (after substituting $\frac{P_1}{P_2} = \frac{\sin 30^0}{\sin 20^0}$)

Alternatively, by parallelogram law of forces

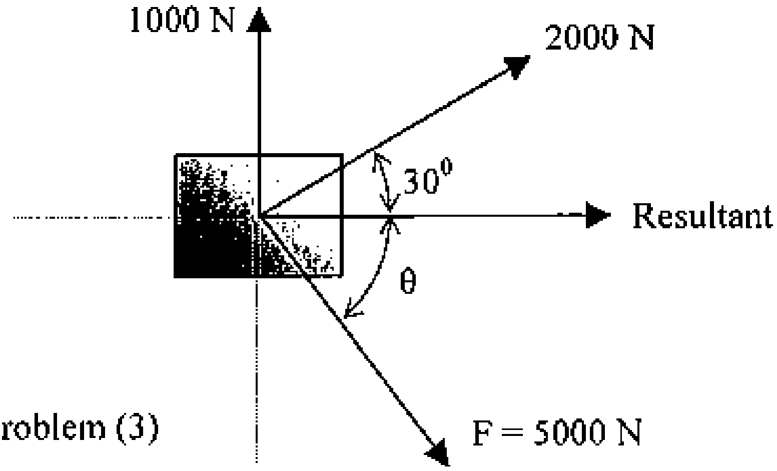
$$\Rightarrow R = \sqrt{P_1^2 + P_2^2 + 2P_1P_2\cos \widehat{P_1P_2}} = \sqrt{P_1^2 + P_2^2 + 2P_1P_2\cos(20^0 + 30^0)}$$

substituting $\frac{P_1}{P_2} = \frac{\sin 30^0}{\sin 20^0}$, $R = 1.532088886\ P_1$ or $2.239764114\ P_2$

Answer (2): Ratio of $\frac{P_1}{P_2} = 1.4619$; Resultant (P) = $1.53209\ P_1 = 2.23976\ P_2$

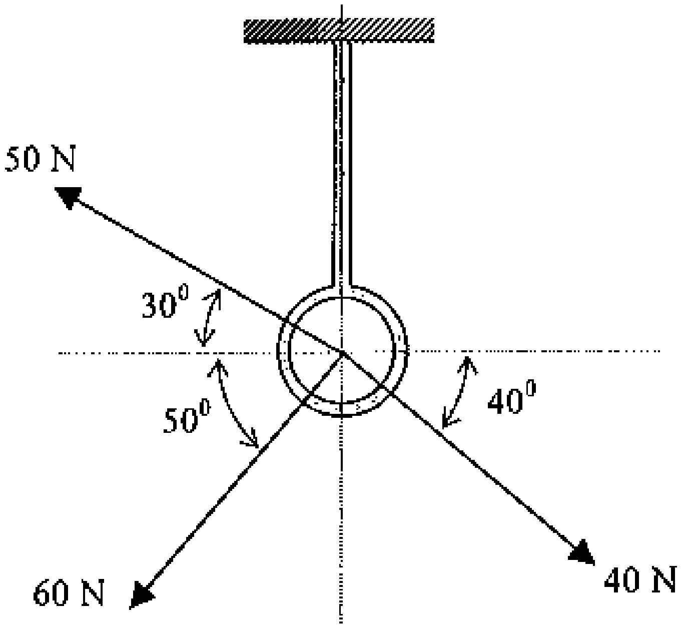
Problem (3): a body is subjected to the three forces. Determine the direction of the force F so that the resultant is in X-direction, when F = 5000 N. What will be the resultant?

Answer (3): $\theta = 53.13^0$; resultant = 4732.0508 N (→)



Problem (3)

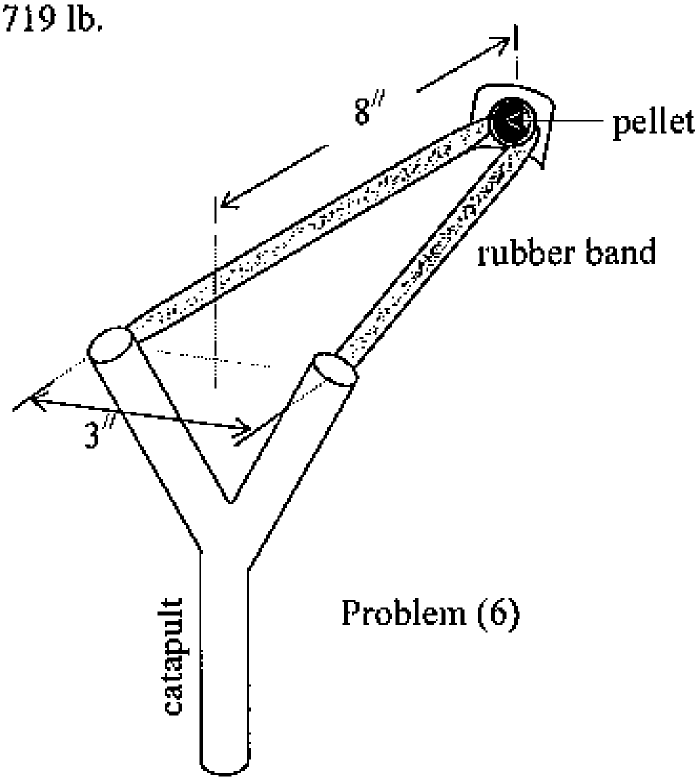
Problem (4): Determine the resultant of the three forces acting on an eye bolt.
Answer (4): F = 69.30128 N; $\theta = 137.663^0$



Problem (4)

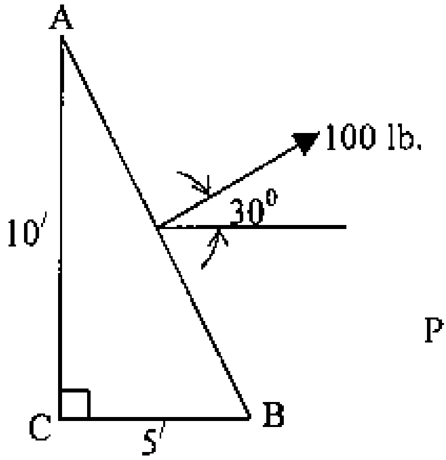
Problem (5): Resolve a 100 lb. force into two components, one making an angle of 30^0 with the force and the other an angle of 75^0 with it.
Answer (5): 100 lb. (30^0 with the 100 lb. force) & 51.76380902 lb. (75^0 with the 100 lb. force).

Problem (6): A simple slingshot about to be "fired". If the rubber band has a stretch of 3 lb. per inch, what force does the band exert on the pellet and hand? The rubber band is 5 inch long when unstreched.
Answer (6): 33.25691719 lb.



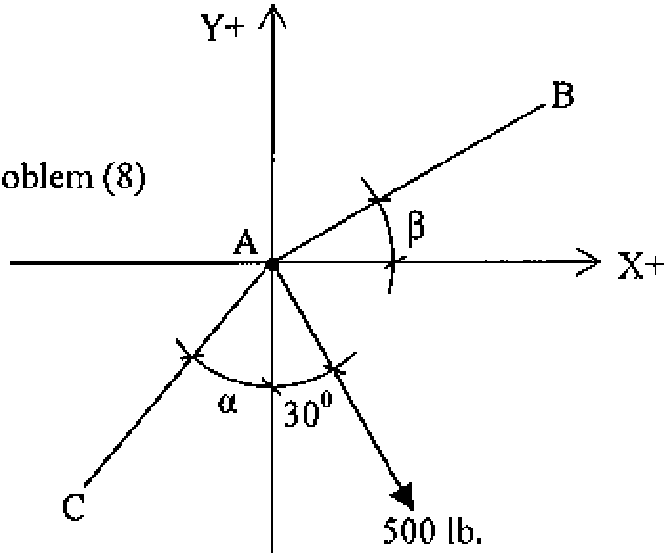
Problem (6)

Problem (7): Resolve the 100 lb. force into components parallel to BC & AB.
Answer (7): $F_{BC} = 111.6025404$ lb. & $F_{AB} = 55.90169945$ lb.



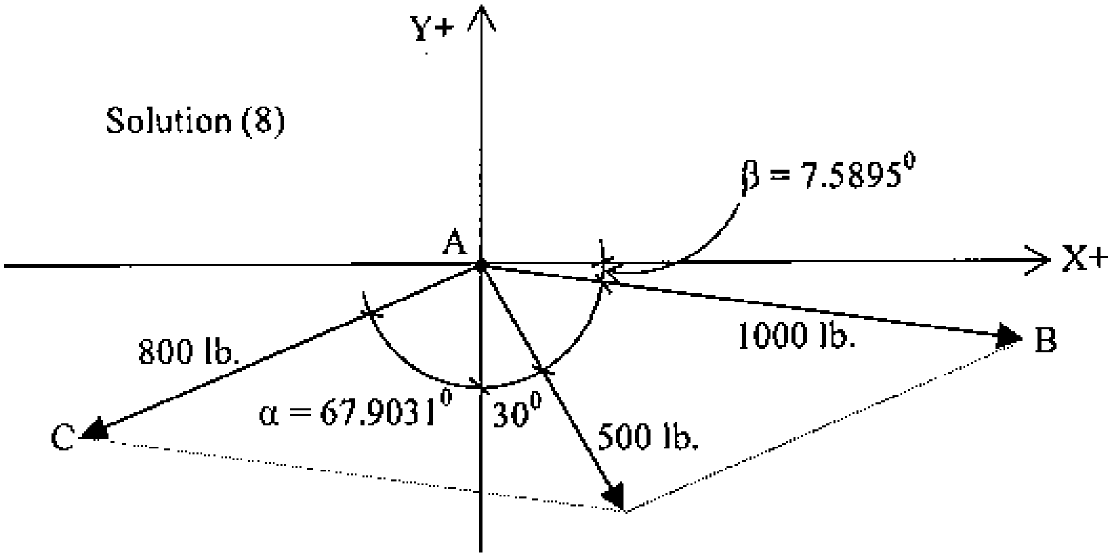
Problem (7)

Problem (8): The 500 lb. force is to be resolved into components along the AB & AC directions measured by the angles α & β. If the component along AC is to be 800 lb. & the component along AB is to be 1000 lb., compute α & β.



Problem (8)

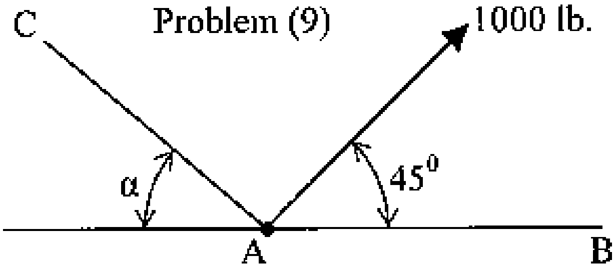
Solution (8): Resultant = 500 lb. & component forces = 800 lb. & 1000 lb.
Resultant in X-direction = ∑components forces in X direction.
» $5 \cos 60^0 = 10 \cos \beta - 8 \sin \alpha$ .... (i)
Resultant in Y-direction = ∑components forces in Y direction.
» $5 \sin 60^0 = 8 \cos \alpha - 10 \sin \beta$ .... (ii)
Solving equations (i) & (ii); $\alpha = 67.9032089^0$ & $\beta = -7.589501548^0$.



Solution (8)

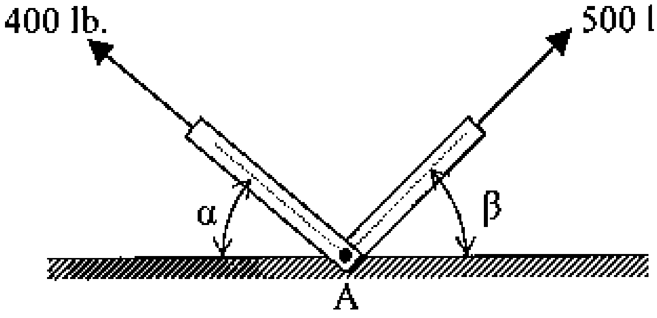
Answer (8): $\alpha = 67.9032089^0$ & $\beta = -7.589501548^0$.

Problem (9): A 100 lb. force is resolved into components along AB & AC. If the component along AB is 700 lb., determine the angle α and the value of the component along AC.
Answer (9): $\alpha = 89.42416779^0$ & $F_{AC} = 707.1424937$ lb.



Problem (9)

Problem (10): Forces are transmitted by two members to pin A. if the sum of these forces is 700 lb. directed vertically, what are the angles α & β.
Answer (10): $\alpha = 55.95226763^0$ & $\beta = 45.5846914^0$.

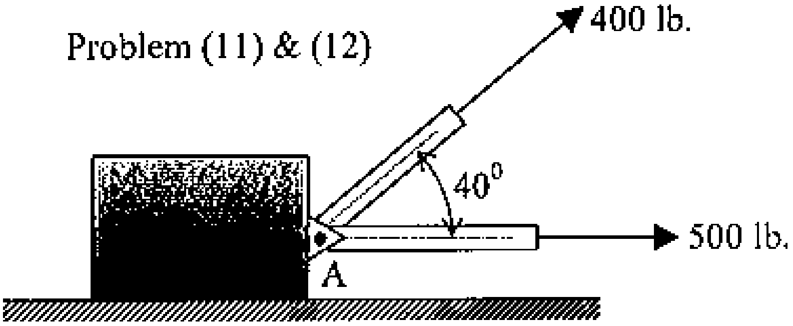


Problem (10)

Problem (11): What is the sum of the forces transmitted by the structural members to the pin at A?
Answer (11): 846.4146603 lb.

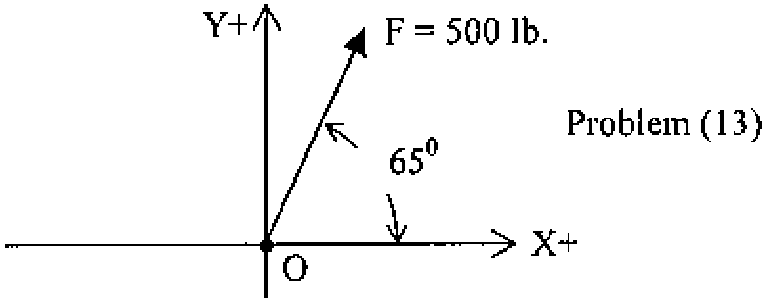
Problem (12): In problem (11), if the total force transmitted by the members to the pin at A be inclined 12^0 to the horizontal & if we do not change the force transmitted by the horizontal member, what must be the new force for the other member? What is the resultant force?

Answer (12): Force in inclined member = 221.431613 lb.; Resultant = 684.5863101 lb.



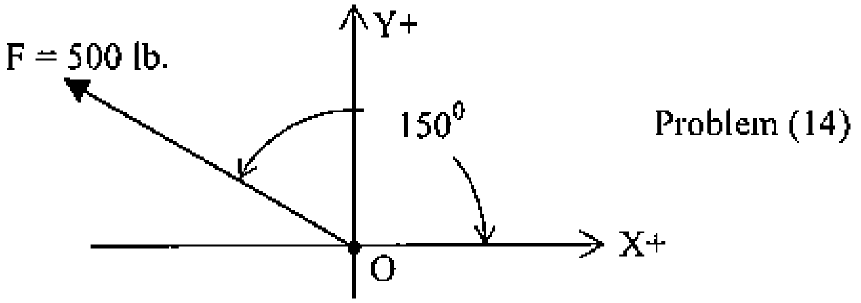
Problem (13): Find the rectangular components F_X and F_y.

Answer (13): $F_X = 211.3091309$ lb. (→) and $F_y = 453.1538935$ lb. (↑).



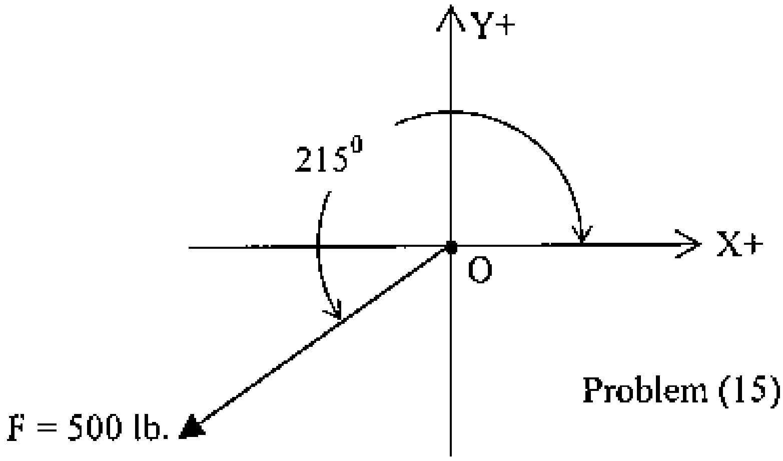
Problem (14): Find the rectangular components F_X and F_y.

Answer (14): $F_X = 433.0127019$ lb. (←) and $F_y = 250$ lb. (↑).



Problem (15): Find the rectangular components F_X and F_y.

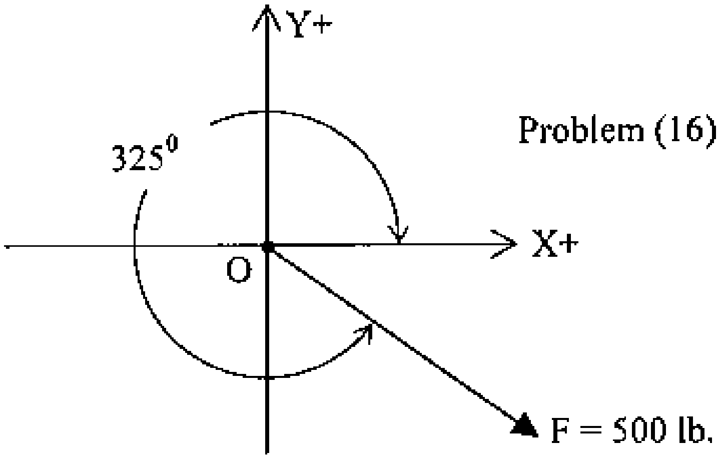
Answer (15): $F_X = 409.5760221$ lb. (←) and $F_y = 286.7882182$ lb. (↓).



Problem (15)

Problem (16): Find the rectangular components F_x and F_y.

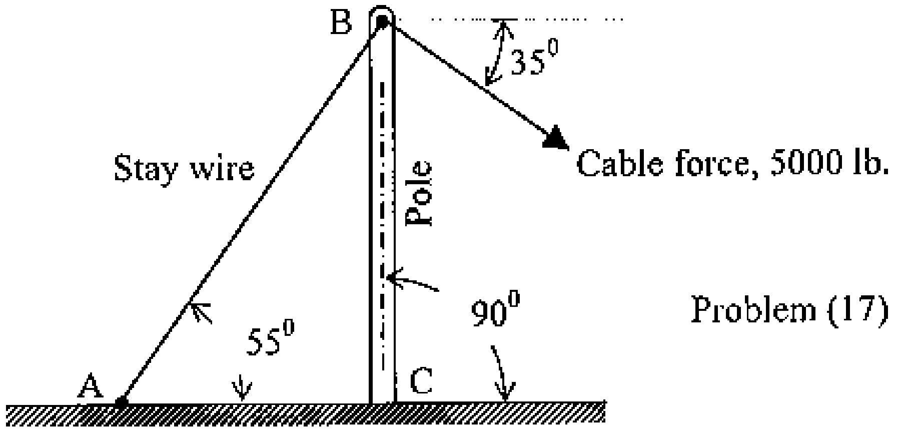
Answer (16): F_x = 409.5760221 lb. (→) and F_y = 286.7882182 lb. (↓).



Problem (16)

Problem (17): A cable exerts 5000 lb. force on a vertical pole. The force in the stay wire should cause the resultant force on the pole to be downward and collinear with the pole. Determine (a) the horizontal & vertical components of the cable pull, (b) the horizontal & vertical components of the reaction in the stay wire & (c) the net force on the pole.

Answer (17): (a) 4095.760221 lb. (→) & 2867.882182 lb. (↓) (b) 4095.760221 lb. & 5849.351796 lb. and (c) 8717.233978 lb.



Problem (17)

Chapter Three

Coplanar Concurrent Forces

If all the forces in a system lie in a single plane and pass through a single point, then the system constitutes a coplanar concurrent force system.

If the resultant of a number of forces, acting on particle is zero, the particle will be in equilibrium. Such a set of forces, whose resultant is zero, are called equilibrium forces.

The force, which brings the set of forces in equilibrium, is called an equilibrant. The equilibrant is equal to the resultant force in magnitude, but opposite in direction.

Resultant of a force system is one which will have the same translation and rotational effect as in the given system of forces. It may be a single force, a system of force through a single point and moment, or a moment only. The concept of rotational effect (moment) need not be known when only the concurrent force systems and handled.

Equilibrium Conditions:

Concurrent forces: (i) The algebraic sum of all the horizontal components of the forces must be equal to zero, i.e. the total sum to the *left* must equal the sum to the *right*.

(ii) The algebraic sum of all the vertical components of the forces must be equal to zero, i.e. the total sum *upwards* must equal the total sum *downwards*.

Triangle Law of Forces: If two forces acting simultaneously on a body are represented by the sides of a triangle taken in order, their resultant is represented by the closing side of the triangle taken in the opposite order.

If more than two forces are acting on a body, two forces at a time can be combined by the triangle law of forces and finally the resultant of all the forces acting on the body can be obtained (see figure 1).

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\overrightarrow{F_{AB}} + \overrightarrow{F_{BC}} = \overrightarrow{F_{AC}}$$

$$\vec{P} + \vec{Q} = \vec{R}$$

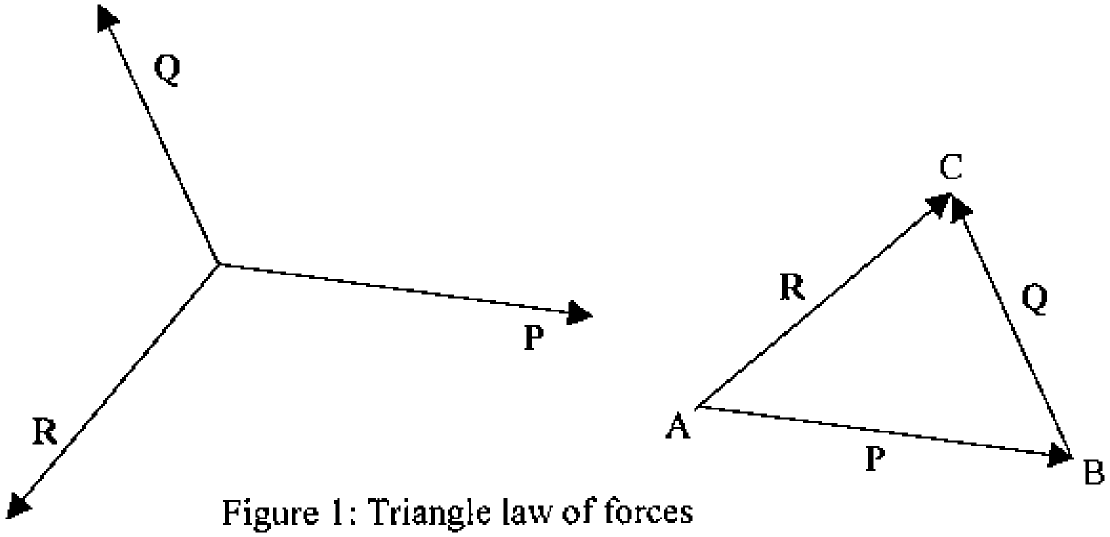


Figure 1: Triangle law of forces

Polygon Law of Forces: If a number of concurrent forces acting simultaneously on a body are represented in magnitude and direction by the sides of a polygon taken in order, then the resultant is represented in magnitude and direction by the closing side of the polygon, taken in the opposite order (see figure 2).

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}=\overrightarrow{AE}$$

$$\overrightarrow{F_{AB}}+\overrightarrow{F_{BC}}+\overrightarrow{F_{CD}}+\overrightarrow{F_{DE}}=\overrightarrow{F_{AE}}$$

$$\vec{P}+\vec{Q}+\vec{R}+\vec{S}=\vec{T}$$

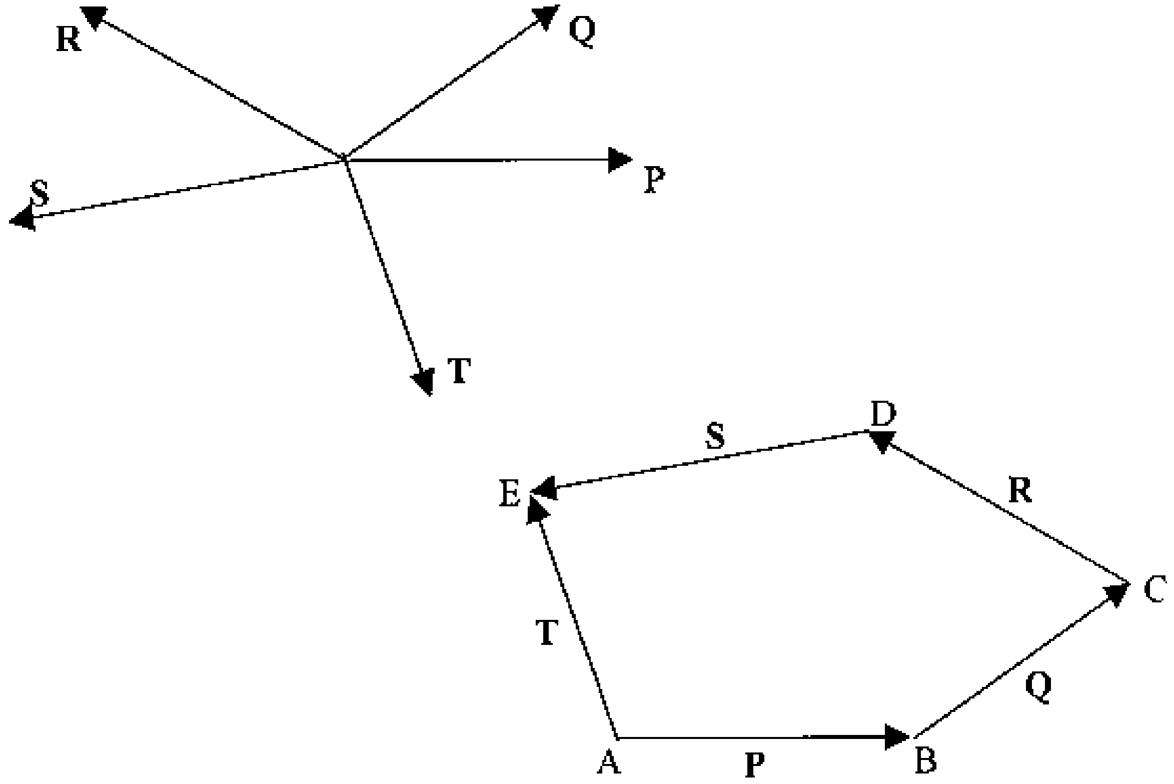


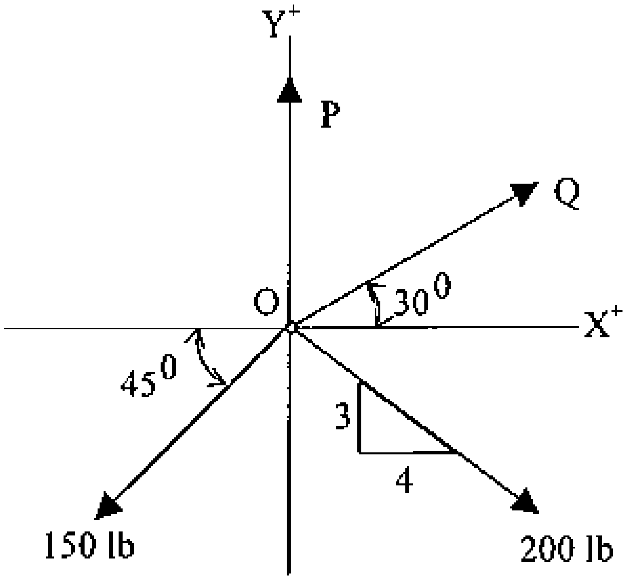
Figure 2: Polygon law forces

Experimental Verification of Triangle Law of Forces: See page no 34 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St. Paul's House Warwick Lane, London EC4.

Experimental Verification of Polygon Law of Forces: See page no 40 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St. Paul's House Warwick Lane, London EC4.

Problems

Problem (1): Determine the magnitudes of P and Q necessary for equilibrium of point O.



Problem (1)

Solution (1): $\sum F_x = 0$; (→+ve)

$$\Rightarrow +Q\cos 30^0 - 150\cos 45^0 + 200 \times \frac{4}{\sqrt{3^2+4^2}} = 0$$

$$\Rightarrow Q = -62.227 \text{ lb.}$$

$\sum F_y = 0$; (↑+ve)

$$\Rightarrow +P + Q\sin 30^0 - 150\sin 45^0 - 200 \times \frac{3}{\sqrt{3^2+4^2}} = 0$$

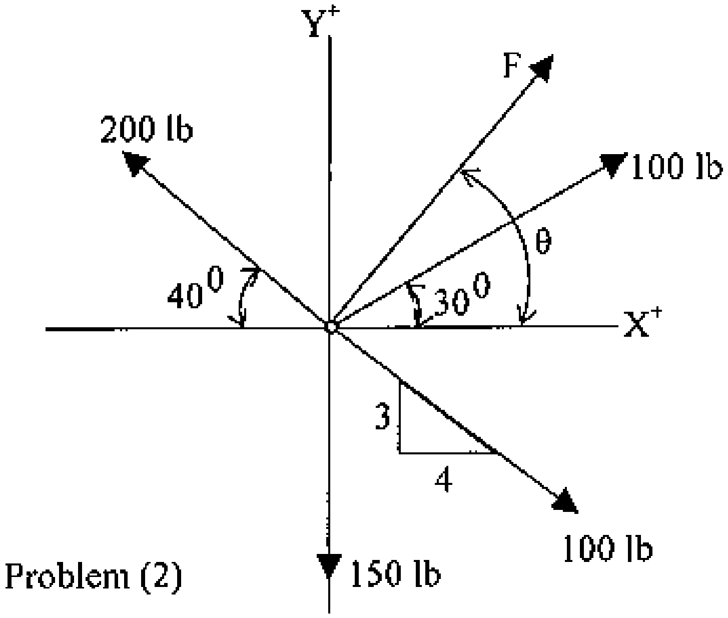
$$\Rightarrow +P = -Q\sin 30^0 + 150\sin 45^0 + 200 \times \frac{3}{\sqrt{3^2+4^2}} = 0$$

$$\Rightarrow +P = -(-62.227)\sin 30^0 + 150\sin 45^0 + 200 \times \frac{3}{\sqrt{3^2+4^2}} = 0$$

$$\Rightarrow P = +257.205 \text{ lb.}$$

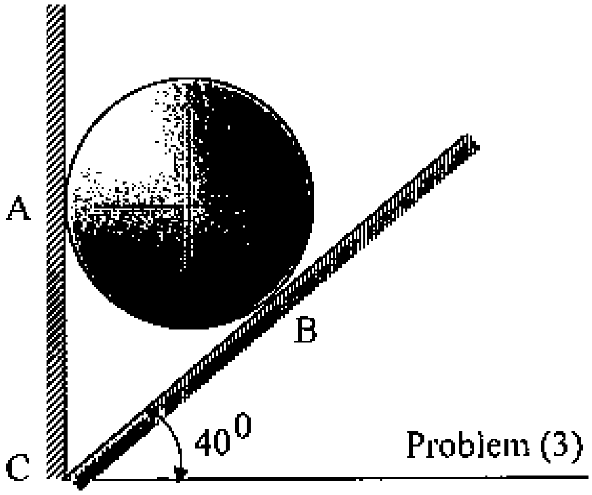
Answer (1): P = + 257.205 lb; Q = – 62.227 lb.

Problem (2): The forces shown in equilibrium. What are F and θ?
Answer (2): F = 119.312 lb; $\theta = 263.554^0$



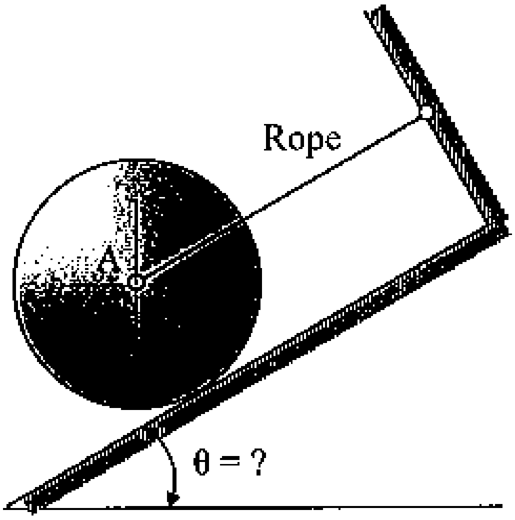
Problem (2)

Problem (3): A 5000-lb sphere rest on a smooth plane inclined at an angle 40^0 with the horizontal and against a smooth vertical wall. What are the reactions at the contact surfaces A and B?
Answer (3): 5000 lb., 7070 lb.



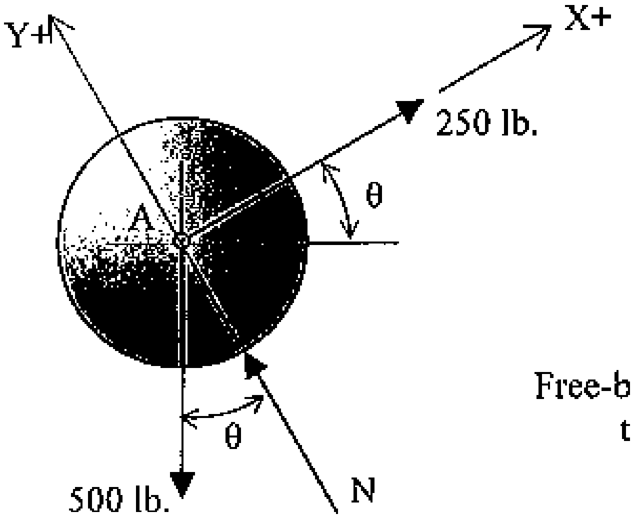
Problem (3)

Problem (4): A 500 lb. cylinder A rest on a smooth inclined plane. For a tension in the rope of 250 lb. find the inclination of the plane and the plane reaction.



Problem (4)

Solution (4):



Free-body diagram of
the sphere

$$\Sigma F_x = 0; +250 - 500\cos\theta = 0$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{250}{500}\right) = 60^0$$

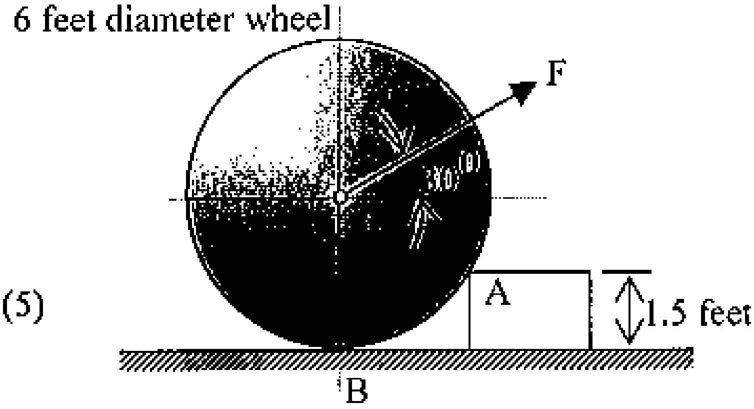
$$\Sigma F_y = 0; +N - 500\sin\theta = 0$$

$$\Rightarrow N = 500\sin\theta = 500\sin(60^0) = 433.0127019 \text{ lb.}$$

Answer (4): 30^0; 433.0127019 lb.

Problem (5): A 2500 lb. wheel with a radius of 3 feet is acted upon by a force F, which tends to pull the wheel over the obstruction at A. At the instant the wheel is about to move, the reaction between the wheel and the ground is zero. What is the magnitude of force F at this instant?

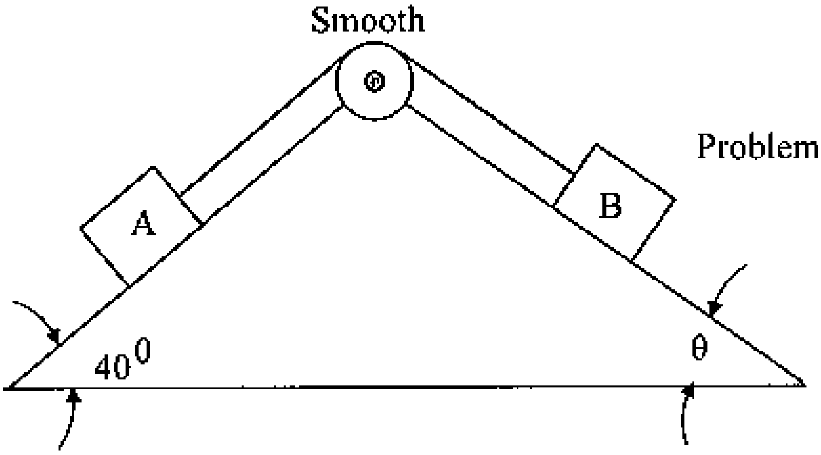
Answer (5): 2500 lb.



Problem (5)

Problem (6): The bodies, A (weight 50 pound) and B (weight 75 pound), are connected by a chord and rest on smooth inclined planes. What is the angle θ if the bodies are in equilibrium? Find also the planes reactions and tension in the chord.

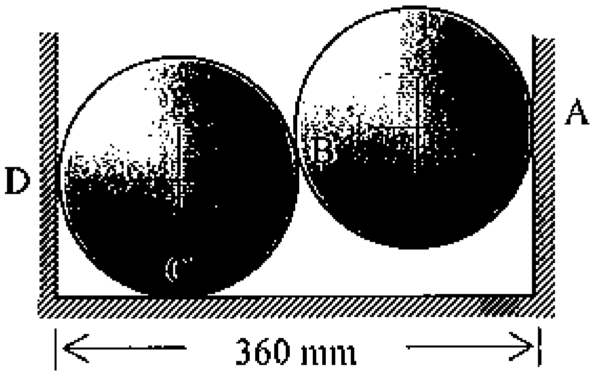
Answer (6): θ = 23.196^0; reaction at A = 38.302 lb; reaction B = 68.937 lb; tension in the chord = 32.139 lb.



Problem (6)

Problem (7): Two smooth spheres, each of radius 100 mm and weighing 100 N, rest in a horizontal channel having vertical walls, the distance between which is 360 mm. find the reactions at the points of contact of A, B, C & D.

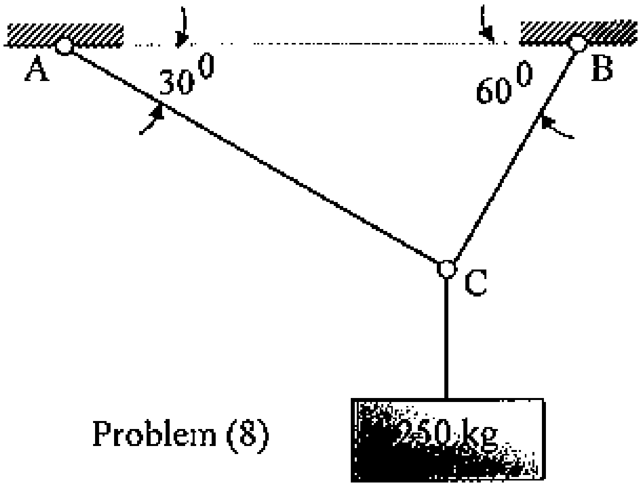
Answer (7): R_A = 133.33 N; R_B = 166.67 N; R_C = 200 N; R_D = 133.33 N



Problem (7)

Problem (8): A 250 kg mass is suspended by flexible cables. Determine the tension in the cables AC and BC.

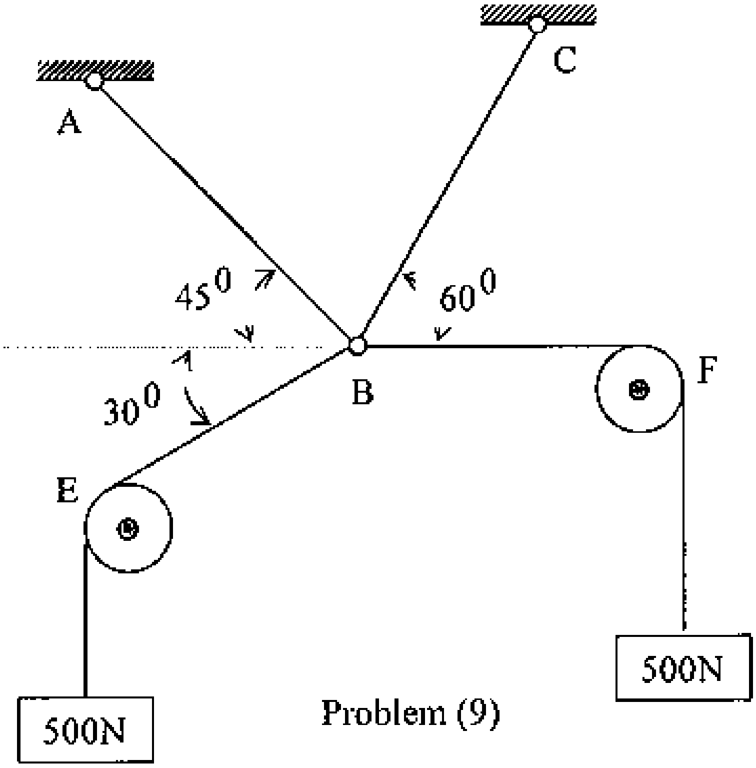
Answer (8): T_{AC} = 125 kg & T_{BC} = 216.50 kg



Problem (8)

Problem (9): Find the tensile forces in cables AB, BC, BE and BF. The pulleys E & F are frictionless.

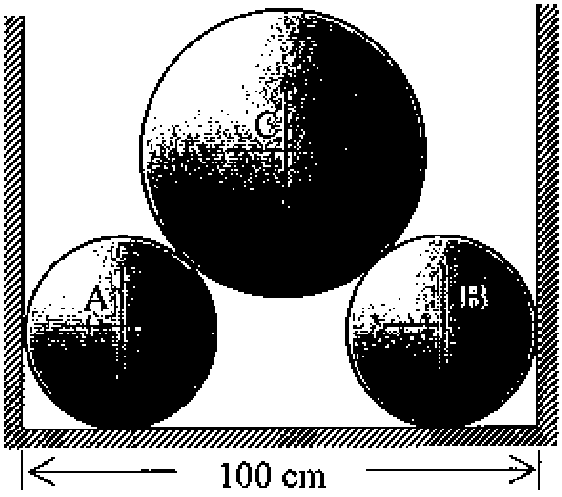
Answer (9): F_{AB} = 189.468 N (tension); F_{BC} = 133.975 N (tension); F_{BE} = F_{BF} = 500 N (tension)



Problem (9)

Problem (10): Cylinders A (diameter 30 cm) and B (diameter 30 cm) have a mass of 200 kg each & cylinder C (diameter 60 cm) has a mass of 400 kg. Compute all contact forces and plane reactions.

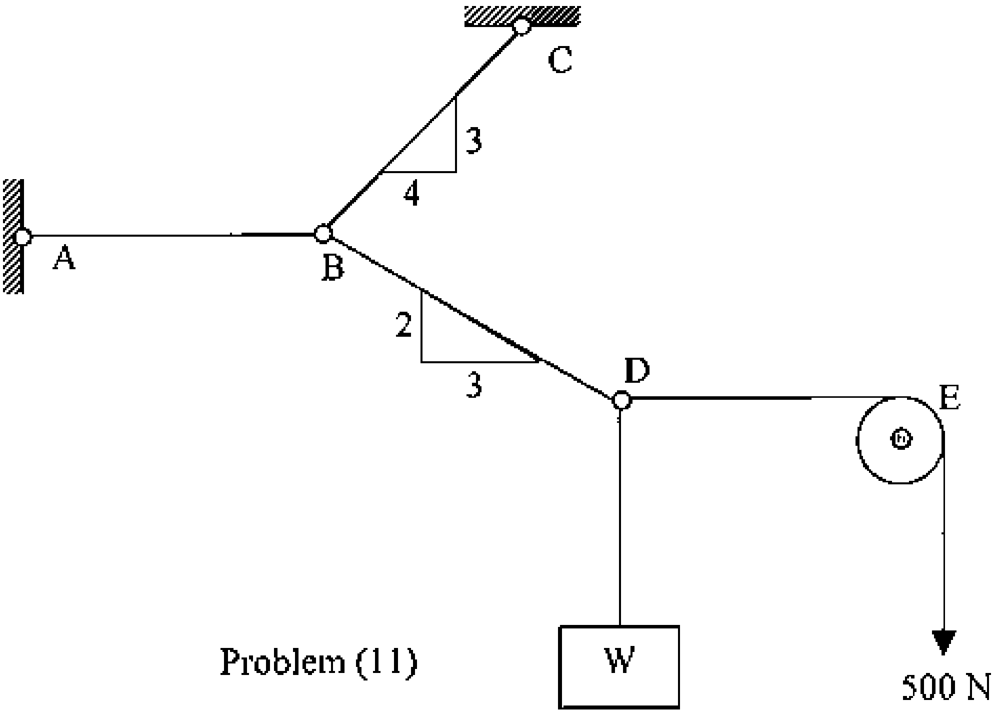
Answer (10): Reactions between cylinder A & C = 318.201 kg; reactions between cylinder B & C = 318.201 kg; reactions between cylinder B & C = 318.201 kg; reactions between cylinder A & horizontal plane = 400 kg; reactions between cylinder B & horizontal plane = 400 kg; reactions between cylinder A & vertical plane = 247.491 kg; reactions between cylinder B & vertical plane = 247.491 kg;



Problem (10)

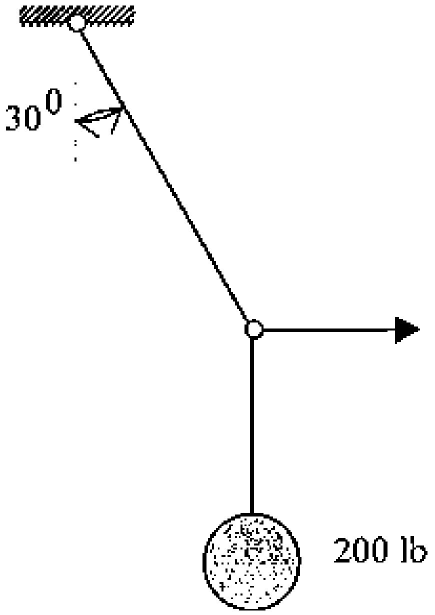
Problem (11): Find the weight W & tensile forces in cables AB, BC, BD & DE.

Answer (9): W = 333.333 N (tension); F_{AB} = 944.444 N (tension); F_{BC} = 555.555 N (tension); F_{BD} =600.925 N (tension); F_{DE} = 500 N (tension)



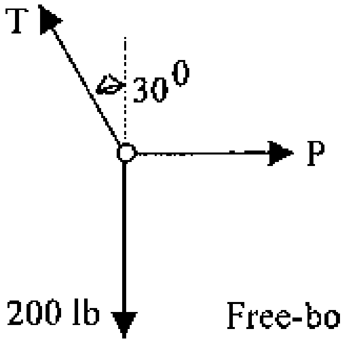
Problem (11)

Problem (12): A weight of 200 lb suspended by a string is pulled aside by a horizontal force P. what is the horizontal force P and the tension in the string?



Problem (12)

Solution (12): Tension in the cable = T



Free-body diagram

Summing the forces in vertical direction (↑ +ve)

$\Rightarrow$ $T \cos 30^0 - 200 = 0$

$\Rightarrow$ $T = 230.9401077$ lb.

Summing the forces in horizontal direction (→ +ve)

$\Rightarrow$ $P \cos 0^0 - T \sin 30^0 = 0$

$\Rightarrow$ $P = T \sin 30^0 = 230.9401077 \sin 30^0 = 115.4700538$ lb.

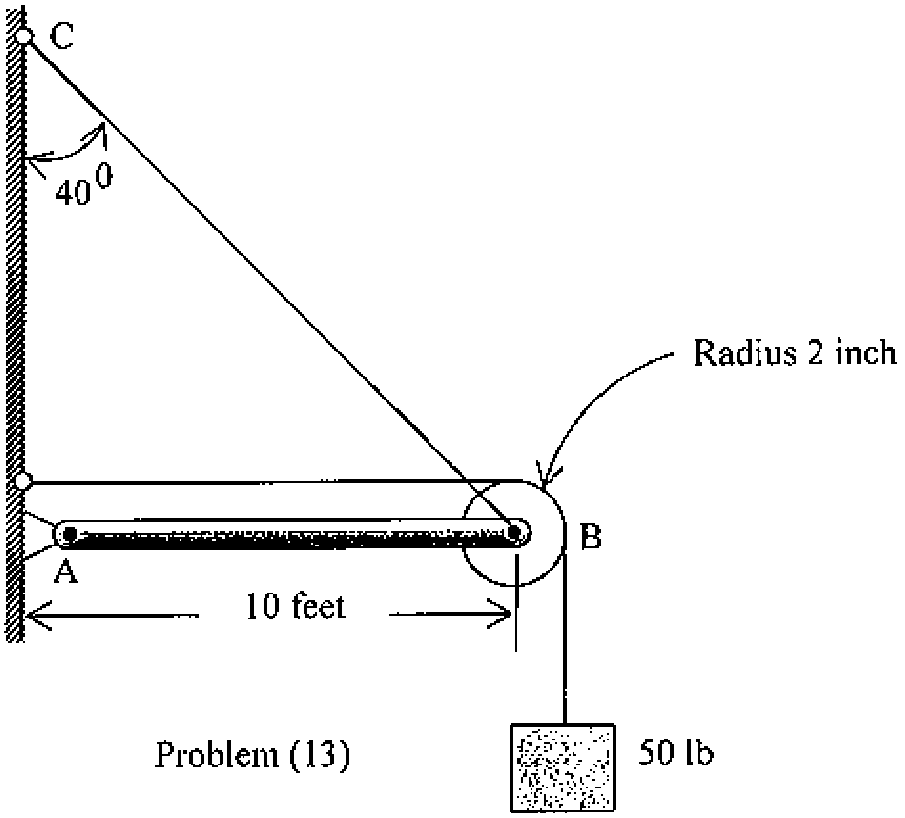
Alternatively, since three-concurrent forces is in equilibrium, Lami's theorem can be applied

$$\frac{T}{\sin 90^0} = \frac{P}{\sin 150^0} = \frac{200}{\sin 120^0}$$

$\Rightarrow$ $T = 230.9401077$ lb. & $P = 115.4700538$ lb.

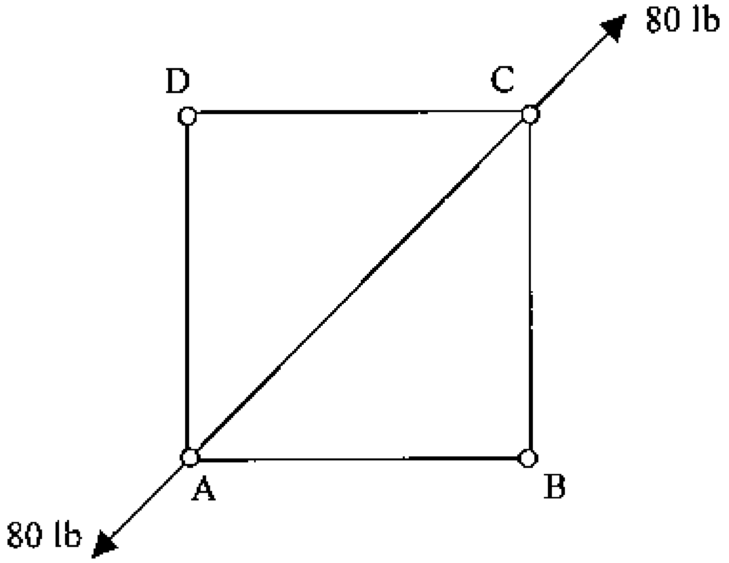
Answer (12): Horizontal force (P) = 115.4700538 lb; Tension in the string = 230.9401077 lb.

Problem (13): Calculate the force in members AB and BC.
Answer (13): $F_{AB} = 91.955$ lb (tension); $F_{BC} = 65.270$ lb (tension)



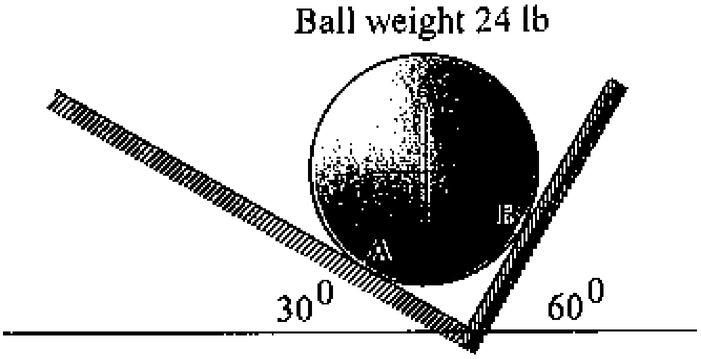
Problem (13)

Problem (14): The square formed by five members is hinged at all corners. Calculate the forces in all the bars.
Answer (14): $F_{AD} = F_{AB} = F_{BC} = F_{CD} = 0$ & $F_{AC} = 80$ lb



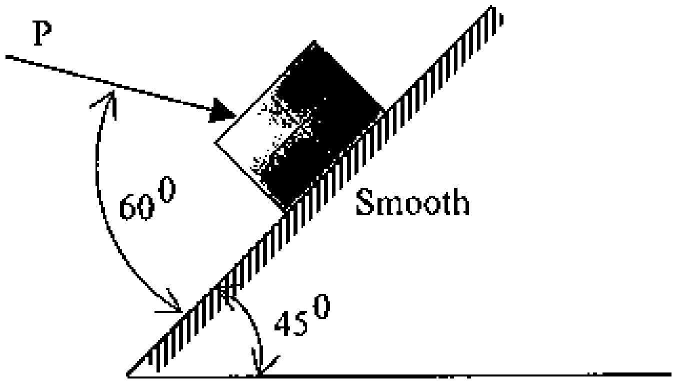
Problem (14)

Problem (15): What are the reactions at point of contact A and B? All the surfaces are assumed smooth.
Answer (15): Reaction at A = 20.8 lb & reaction at B = 12 lb.



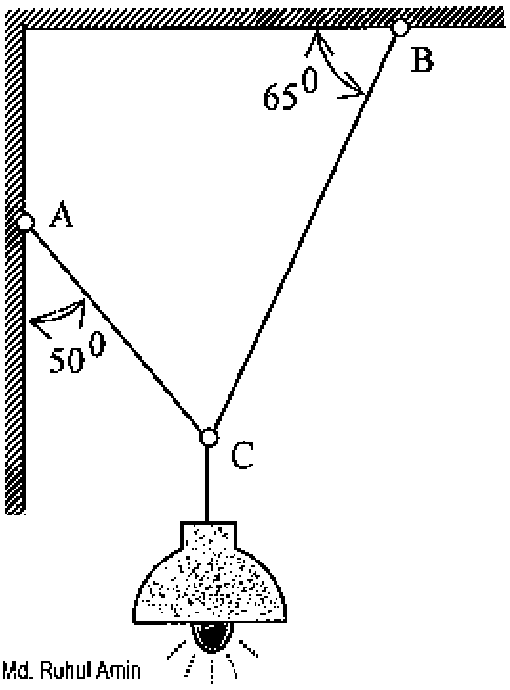
Problem (15)

Problem (16): What force P is required to hold the block (weight 150 lb) stationary?
Answer (16): Force (P) = 212 lb.



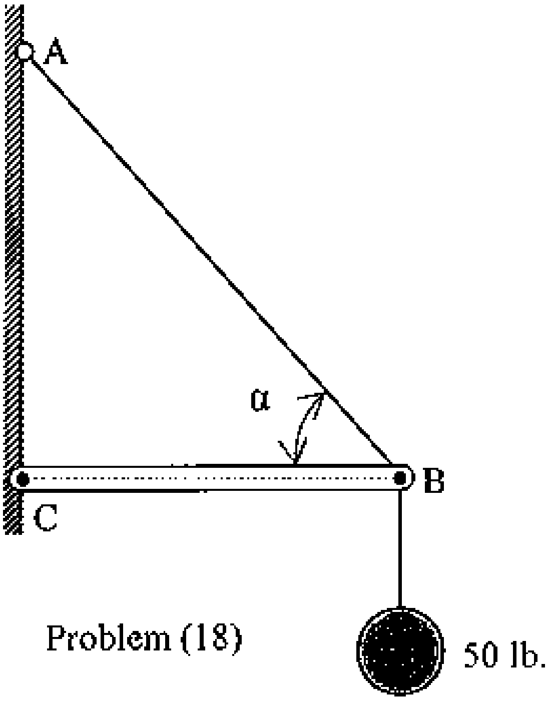
Problem (16)

Problem (17): The electric light fixture has a weight of 50 lb. Determine the tensile force in members AC and BC.
Answer (17): F_{BC} = 39.6 lb, F_{AC} = 22.2 lb.



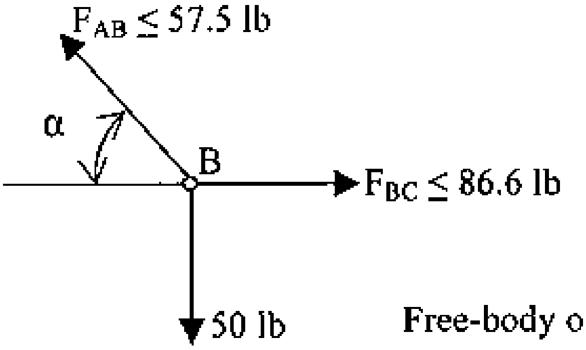
Problem (17)

Problem (18): The weight W is 50 lb., member AB can sustain load of 57.5 lb tension and BC can sustain a load of 86.6 lb compression. Assume all the members are pin connected. What is the angle α?



Problem (18)

Solution (18): Given, F_{AB} (maximum) = 57.5 lb. & Given, F_{BC} (maximum) = 86.6 lb.



Free-body of joint B

Summing the forces in horizontal direction (→ +ve)

$\Rightarrow F_{BC} \cos 0^0 - F_{AB} \cos\alpha = 0$

$\Rightarrow F_{BC} = F_{AB} \cos\alpha$

Summing the forces in vertical direction (↑ +ve)

$\Rightarrow F_{AB} \sin\alpha - 50 = 0$

$\Rightarrow F_{AB} \sin\alpha = 50$

1st condition - if $F_{BC} = 86.6$ lb.

$\Rightarrow F_{BC} = F_{AB} \cos\alpha = 86.6$ & $F_{AB} \sin\alpha = 50$

$$\frac{F_{AB} \sin\alpha}{F_{AB} \cos\alpha} = \tan\alpha = \frac{50}{86.6}; \ \alpha = 48.39646977^0$$

Check, force in member AB, $F_{AB} = \sqrt{(F_{AB}\sin\alpha)^2 + (F_{AB}\cos\alpha)^2} = \sqrt{(50)^2 + (86.6)^2} = 99.99779998$ lb. (which is more than F_{AB} (maximum) = 57.5 lb.) Not OK.

2nd condition - if $F_{AB} = 57.5$ lb.

$\Rightarrow F_{AB}\sin\alpha = 50 = 57.5\sin\alpha$

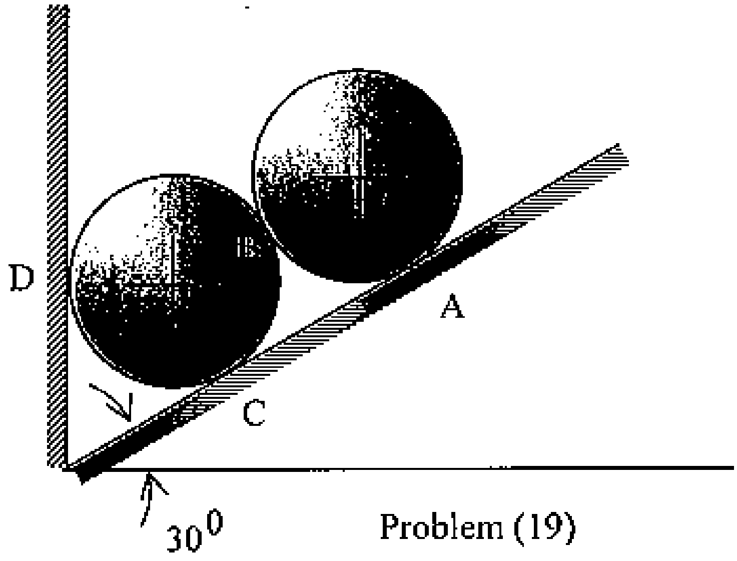
$\Rightarrow \sin\alpha = \frac{50}{57.5}$, $\alpha = 60.40815421^0$

Check, force in member BC, $F_{BC} = F_{AB}\cos\alpha = 57.5\cos 60.40815421^0 = 28.39454173$ (which is less than F_{BC} (maximum) = 86.6 lb.) OK.

Answer (18): $\alpha = 60.40815421^0$

Problem (19): Two identical rollers, each weighing 100 N, are supported by an inclined plane and a vertical wall. Assuming smooth surfaces, find the reactions induced at the points of contact A, B, C and D.

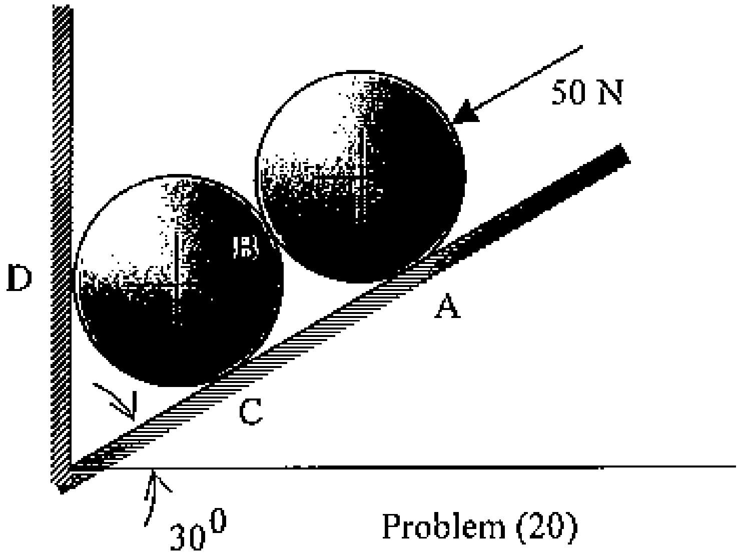
Answer (19): $R_A = 86.60$ N; $R_B = 50$ N; $R_C = 144.34$ N; $R_D = 115.47$ N



Problem (19)

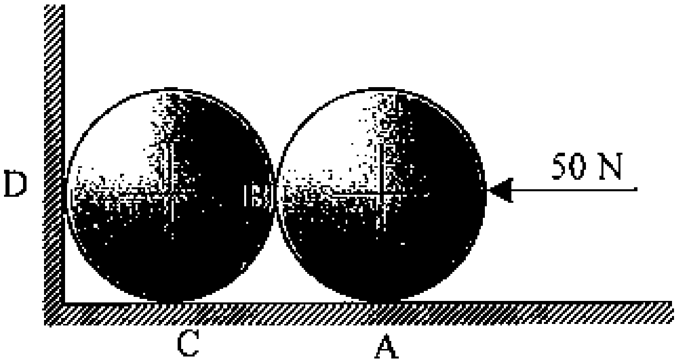
Problem (20): Same as problem (19) with an additional 50 N force.

Answer (20): $R_A = 86.60254038$ N; $R_B = 100$ N; $R_C = 173.2050808$ N & $R_D = 173.2050808$ N.



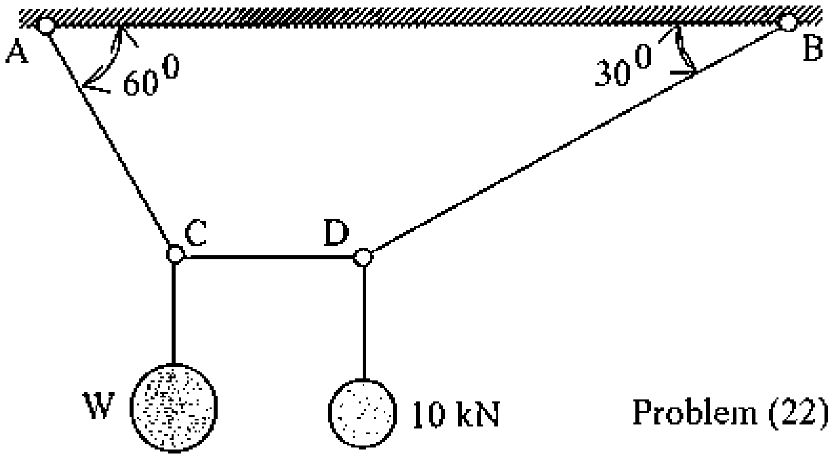
Problem (20)

Problem (21): Same as problem (20) but angle of inclination 0^0 instead of 30^0.
Answer (21): $R_A = 100$ N; $R_B = 50$ N; $R_C = 100$ N; $R_D = 50$ N



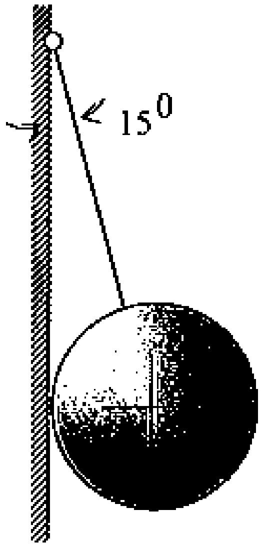
Problem (21)

Problem (22): A chord supported at A and B carries a load of 10 kN at D and a load W at C. Find the value of W so that CD remains horizontal.
Answer (22): W = 30 kN



Problem (22)

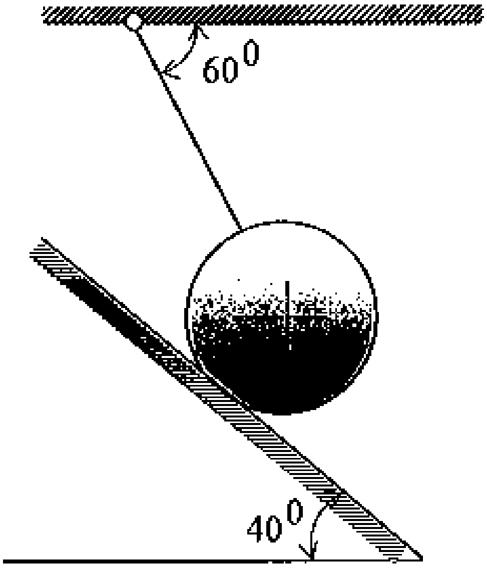
Problem (23): A sphere weighing 100 N is tied to a smooth wall by a string. Determine the tension in the string and the reaction from the wall.
Answer (23): Tension in the string = 103.53 N and Reaction = 26.79 N.



Problem (23)

Problem (24): A ball weighing 500 N is tied to a roof slab and is resting on a plane at 40^0 to the horizontal. Determine the reaction from the inclined plane and the tension in the string.

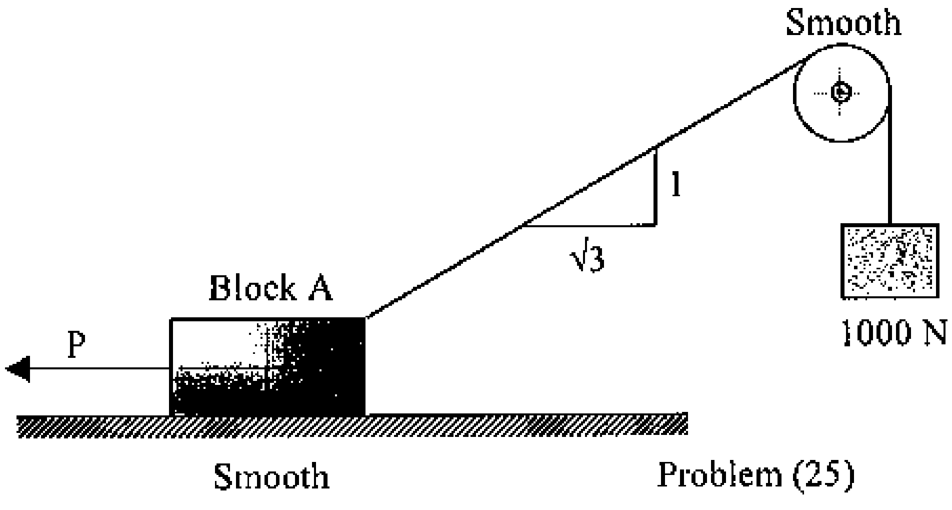
Answer (24): Reaction = 266.04 N & Tension = 342.03 N.



Problem (24)

Problem (25): What force P is needed to resist the downward motion? Find also the reaction of the surface. Given, block A weight = 2500 N

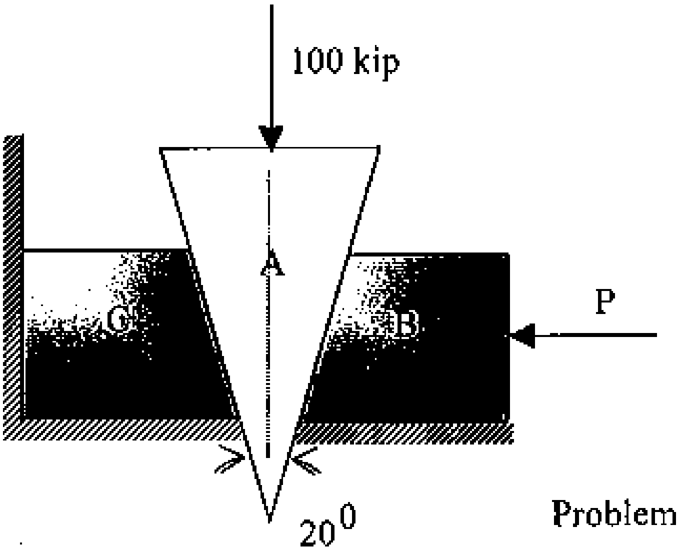
Answer (25): Force (P) = 866.025 N; Reaction = 2000 N



Problem (25)

Problem (26): Wedge A (weight 150 kip) supporting a downward load of 100 kip is pressed between two blocks B (weight 75 kip) and C (weight 50 kip). What P is needed to resist the downward motion of wedge A. Find also the reactions from the planes and contact surfaces. All the surfaces are smooth.

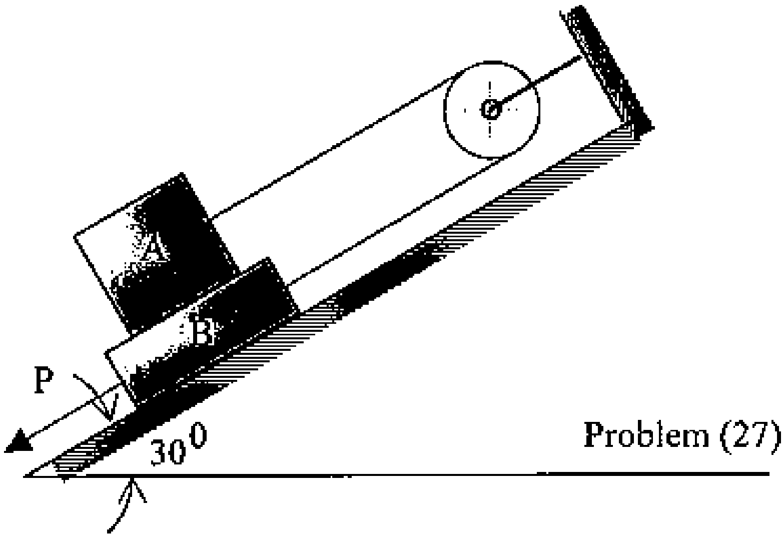
Answer (10): Force (P) = 708.909 kip; reaction between block A & B = 719.846 kip; reaction between block A & C = 719.846 kip; reaction between block C & horizontal plane = 175 kip; reactions between block C & vertical plane = 708.909 kip; reactions between block B & vertical plane = 200 kip.



Problem (26)

Problem (27): Block A (weight 75 lb) and block B (weight 50 lb), rest on each other, connected by a chord which passes over a frictionless pulley. The surfaces between block A & B and block B & 30^0 inclined plane are smooth. What force P is needed to resist the downward motion of block A. Find also the tension in the chord and the reaction between block & A and block B & the 30^0 inclined plane

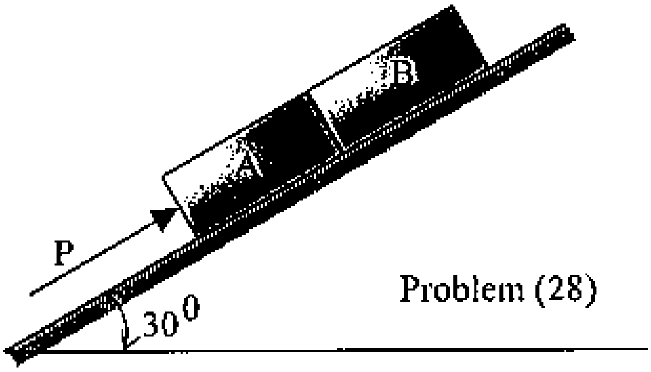
Answer (27): Force (P) = 12.5 lb; tension in the chord = 37.50 lb; reaction between A & B = 64.952 lb; reaction between B & 30^0 inclined plane = 108.253 lb.



Problem (27)

Problem (28): Two resting against each other are placed on a surface inclined at an angle 30^0 to the horizontal. Block A, weighing 30 N, is placed down the incline from block B, which 500 N. What upward force P will resist the downward motion of the two blocks A and B along the plane. Determine also the reaction at contact surface between A & B and plane reactions.

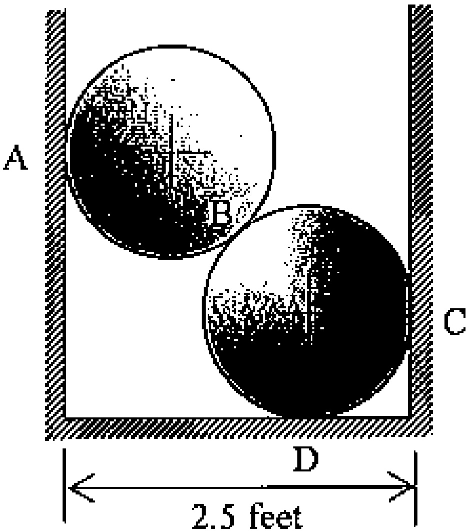
Answer (28): P = 265 N (upward); reaction between A & B = 250 N; reaction between A & 30^0 inclined plane = 25.981 N; reaction between B & 30^0 inclined plane = 433.013 N



Problem (28)

Problem (29): Two smooth cylinders of 2 feet diameter rest in a box. Determine all the contact forces. Each cylinder weighs 50 lb.

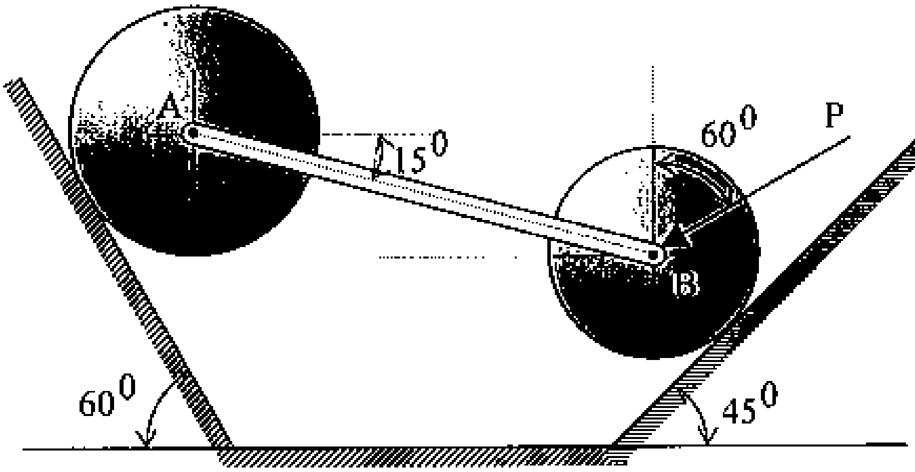
Answer (29): R_A = 12.9103 lb.; R_B = 51.6398 lb.; R_C = 12.9106 lb & R_D =100 lb.



Problem (29)

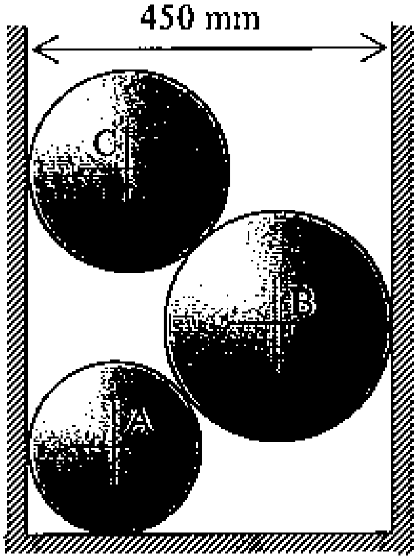
Problem (30): Two cylinders, A of weight 4000 N and B of weight 2000 N rest on smooth inclines. They are connected by a bar of negligible weight hinged to each cylinder at its geometric centre by smooth pins. Find the force P to be applied such that it will hold the system in the given position.

Answer (30): P = 1071.796 N



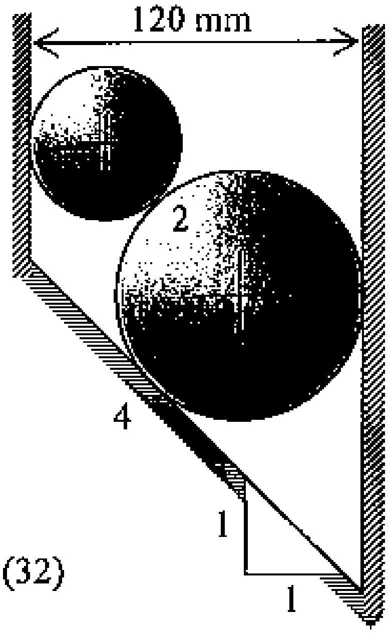
Problem (30)

Problem (31): Three cylinders are pinned in a rectangular ditch. Neglecting friction, determine the reaction between cylinder A and the vertical wall. Given, Cylinder A (weight 75 N, radius 100 mm), Cylinder B (weight 200 N, radius 150 mm), Cylinder A (weight 100 N, radius 125 mm).
Answer (31): 400 N.



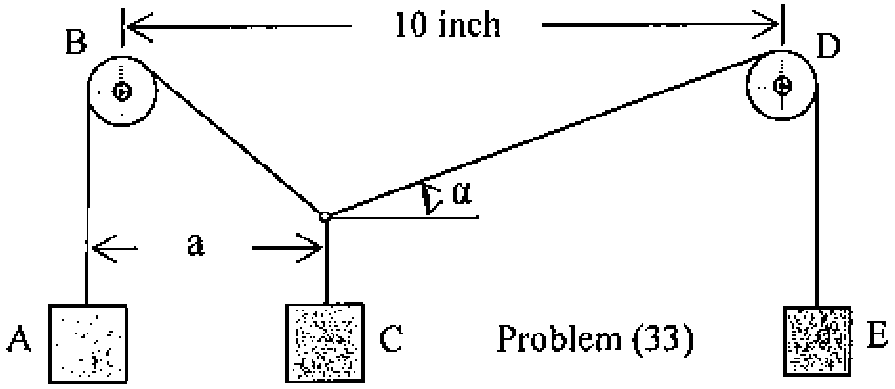
Problem (31)

Problem (32): Two cylinders of diameters 100 mm and 50 mm, weighing 200 N and 50 N, respectively are placed in trough. Neglecting friction, find the reactions at contact surfaces 1, 2, 3 & 4.
Answer (32): $R_1 = 37.5$ N; $R_2 = 62.5$ N; $R_3 = 287.5$ N; $R_4 = 353.5$ N



Problem (32)

Problem (33): A continuous string ABCDE passes over smooth pegs at B & D, 10 inch on centers. To the ends of the string are attached the weights $W_A = 7$ lb and $W_E = 5$ lb. A 10 lb weight is attached C and the bodies are in equilibrium. Determine the distance 'a' and the angle α?

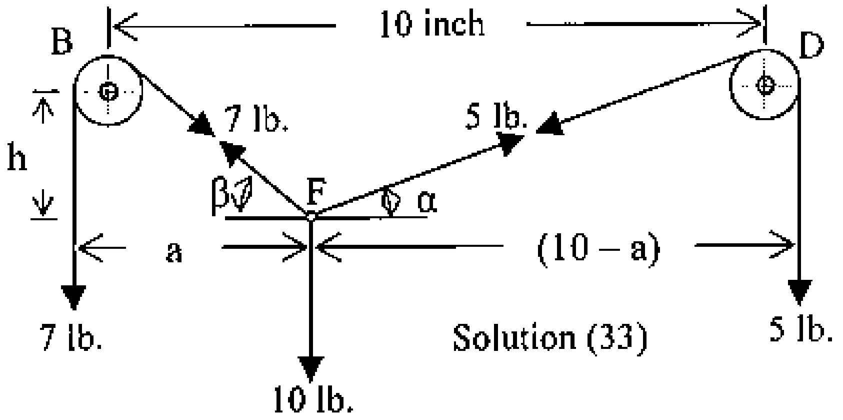


Problem (33)

Solution (33): Free-body of joint F

$$\sum F_x = 0\ (\rightarrow +ve);\ +5\cos\alpha - 7\cos\beta = 0;\ \cos\beta = \frac{5}{7}\cos\alpha$$

$$\Rightarrow \sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{5}{7}\cos\alpha\right)^2} = \frac{\sqrt{49-25\cos^2\alpha}}{7}$$



Solution (33)

$\sum F_y = 0$ (↑+ve); $+5\sin\alpha + 7\sin\beta - 10 = 0$;

$\Rightarrow 7\sin\beta = 10 - 5\sin\alpha$

$\Rightarrow 7 \times \dfrac{\sqrt{49 - 25\cos^2\alpha}}{7} = 10 - 5\sin\alpha$; $\Rightarrow \sqrt{49 - 25\cos^2\alpha} = 10 - 5\sin\alpha$

$\Rightarrow 49 - 25\cos^2\alpha = (10 - 5\sin\alpha)^2 = 100 - 100\sin\alpha + 25\sin^2\alpha$

$\Rightarrow 100\sin\alpha = 100 - 49 - 25(\sin^2\alpha + \cos^2\alpha) = 100 - 49 - 25 = 76$

$\Rightarrow \sin\alpha = \dfrac{76}{100}$; $\alpha = \sin^{-1}\left(\dfrac{76}{100}\right) = 49.46419789^0$.

$>> \cos\beta = \dfrac{5}{7}\cos\alpha$

$\Rightarrow \beta = \cos^{-1}\left(\dfrac{5}{7}\cos\alpha\right) = \cos^{-1}\left(\dfrac{5}{7}\cos 49.46419789^0\right) = 62.3395501^0$.

$>> \tan\beta = \dfrac{h}{a}$ & $\tan\alpha = \dfrac{h}{(10 - a)}$

$\Rightarrow h = a\tan\beta = (10 - a)\tan\alpha$

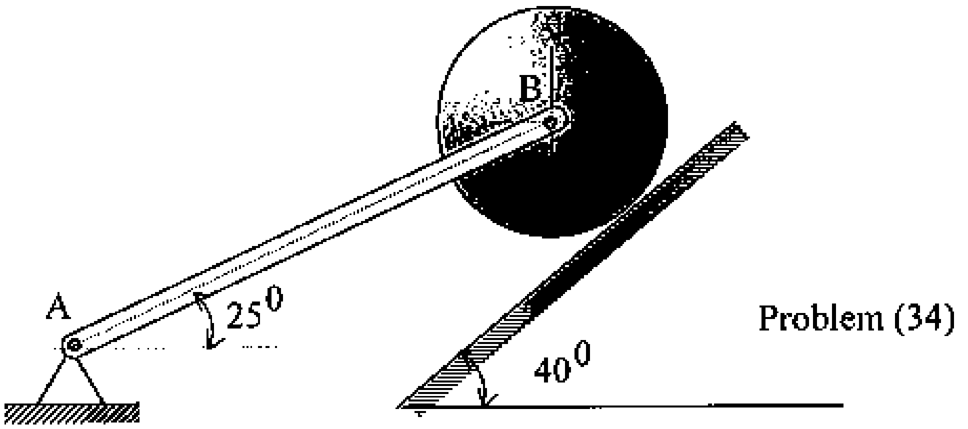
$\Rightarrow a\tan 62.3395501^0 = (10 - a)\tan 49.46419789^0$.

$\Rightarrow a = 3.80$ inch.

Answer (33): a = 3.80 inch; $\alpha = 49.46419789^0$.

Problem (34): Determine the force in the bar AB and plane reaction and reaction components at B. Bar AB is weightless and cylinder has a weight of 50 lb.

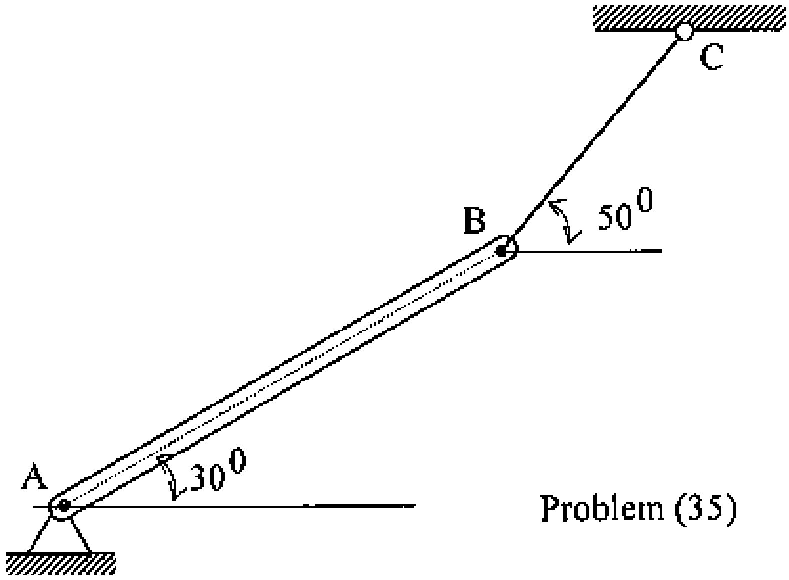
Answer (34): $F_{AB} = 33.2731$ lb (compression); plane reaction = 46.9137 lb; $R_{AX} = 30.1556$ lb (→); $R_{AY} = 14.0618$ lb (↑)



Problem (34)

Problem (35): Determine the tension in the cable BC and reaction components at A. Bar AB has a weight of 20 lb & length of 10 feet.

Answer (35): $T_{BC} = 25.3216$ lb; $R_{AX} = 26.2764$ lb (→); $R_{AY} = 0.6025$ lb (↑)

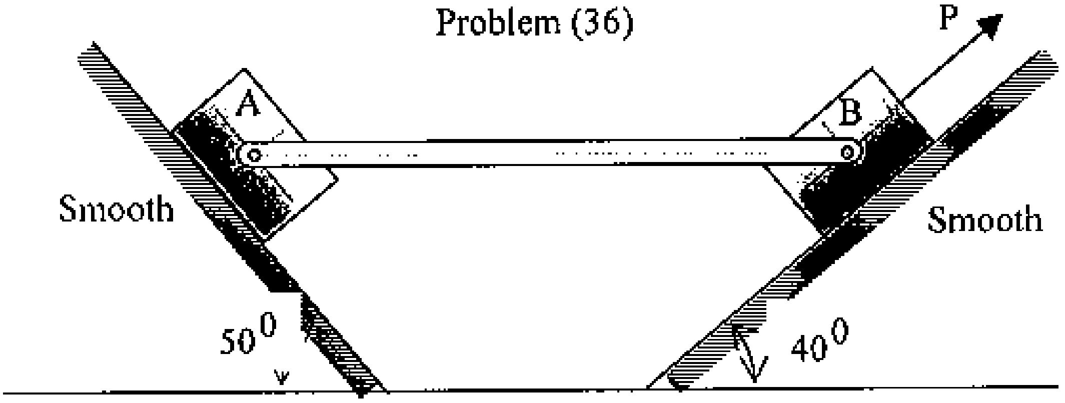


Problem (35)

Problem (36): Two blocks A (weight 100 lb) and B (weight 200 lb) are connected by a horizontal straight bar. What force P required to prevent the downward motion of block A? Find also the plane reactions and bar force. Assume all the planes are smooth.

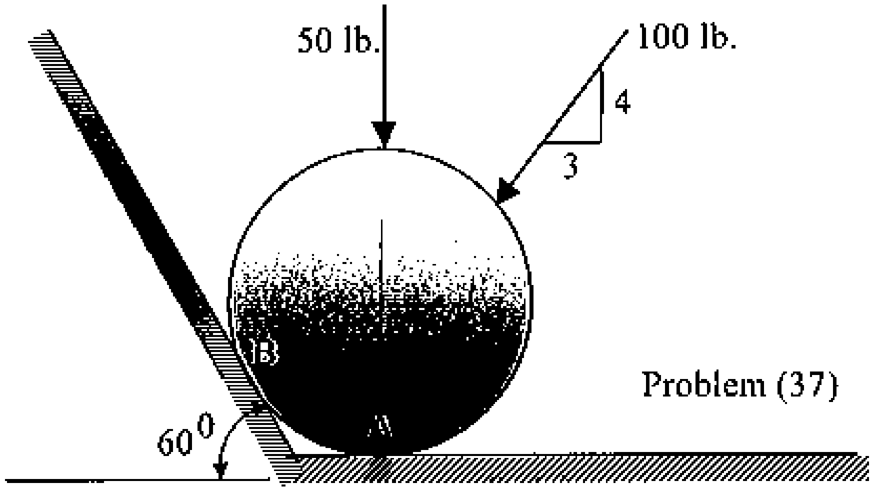
Answer (36): P = 37.26417 lb; R_A=155.572 lb; R_B= 229.813 lb; F_{AB}= 119.175 lb (compression)

Problem (36)



Problem (37): A sphere of weight 50 lb. rest on smooth planes. What will the plane reactions at A & B under the action of forces?

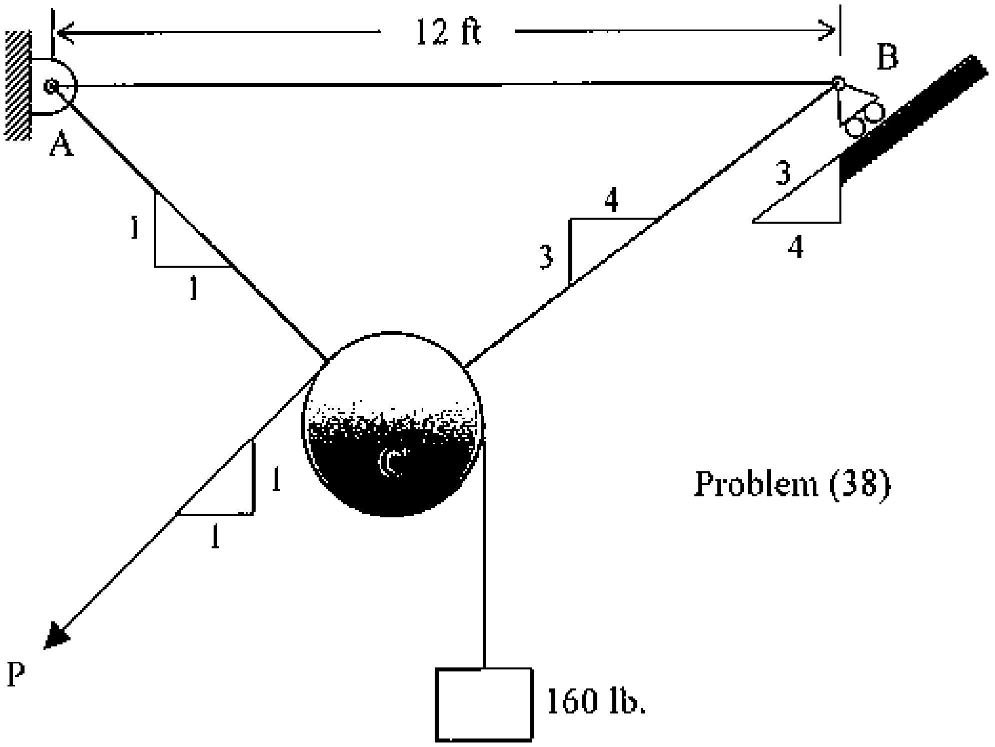
Answer (37): $R_A = 145.3589\,\text{lb}(\uparrow)$; $R_B = 69.2820\,\text{lb}(\nearrow)$



Problem (37)

Problem (38): The frame (see figure) is pin-supported at A and roller supported at B. The pulley is free to rotate on a frictionless pin at C. the weight 100 lb. is held in equilibrium by the force P in the rope. What forces are induced in the members of the frame?

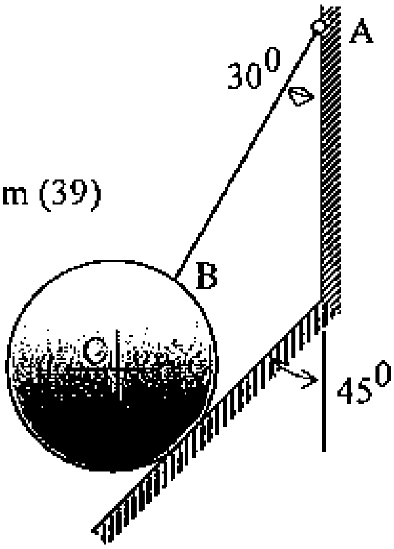
Answer (38): F_{AB} = 344.8876518 lb. (compression); F_{AC} = 152.1566686 lb. (tension) & F_{BC} = 275.9101214 lb. (tension)



Problem (38)

Problem (39): Determine tension in the chord AB. Sphere C weighs 100 lb. & all the contact surfaces are smooth.

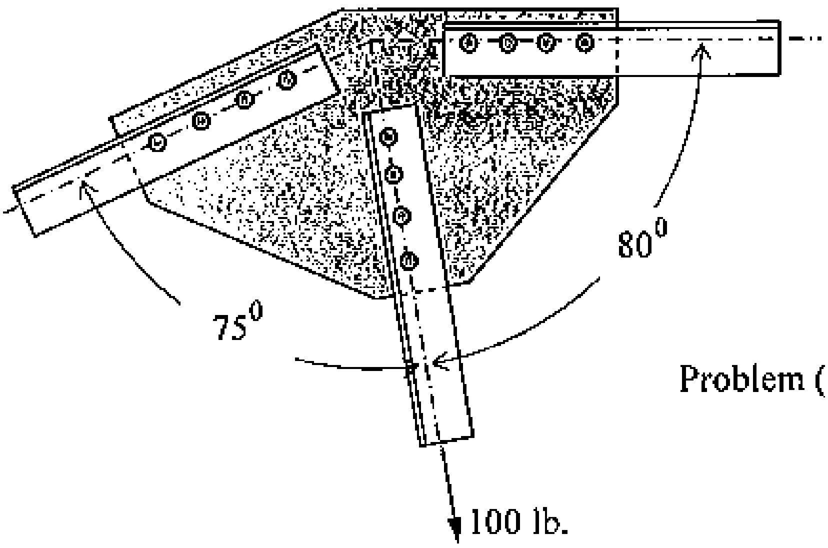
Answer (39): T_{AB} = 73.20508076 lb. (tension)



Problem (39)

Problem (40): Determine the two unknown forces in the truss members. Are they tensile or compressive forces?

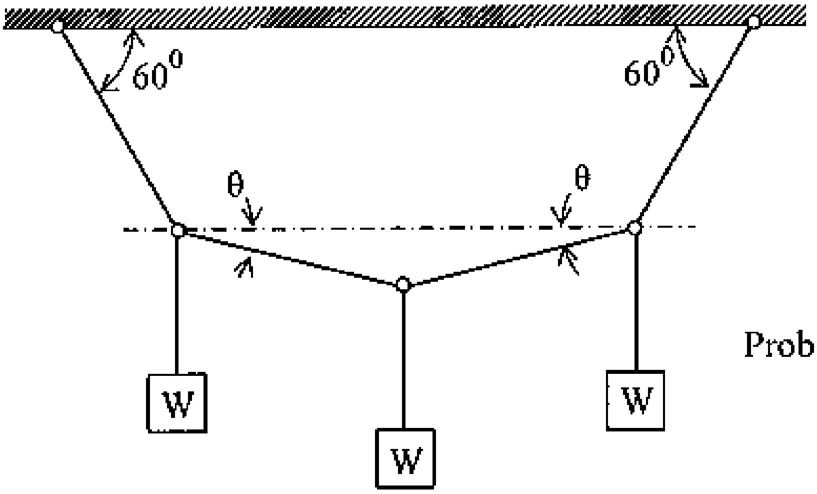
Answer (40): – 228.5575219 lb. (compression) & – 233.0253664 lb. (compression).



Problem (40)

Problem (41): The equal weights are symmetrically placed. Determine the angle θ?

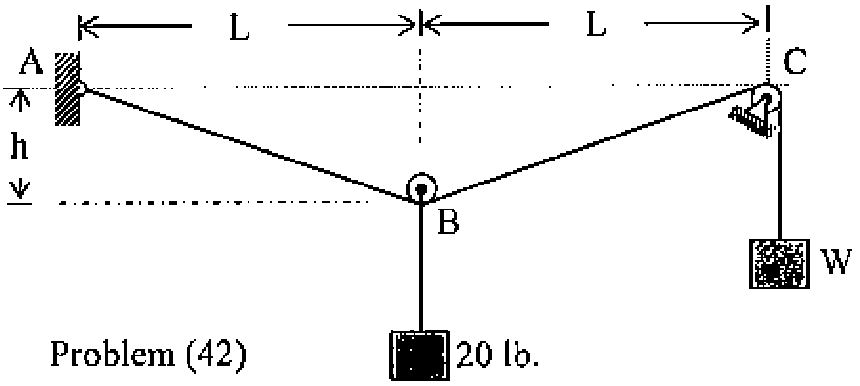
Answer (41): $\theta = 30^0$



Problem (41)

Problem (42): Determine W as a function of h.

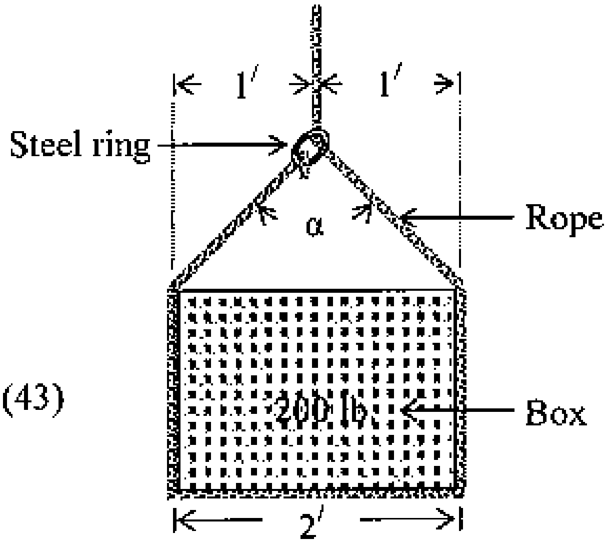
Answer (42): $W = \dfrac{10\sqrt{h^2 + L^2}}{h}$



Problem (42)

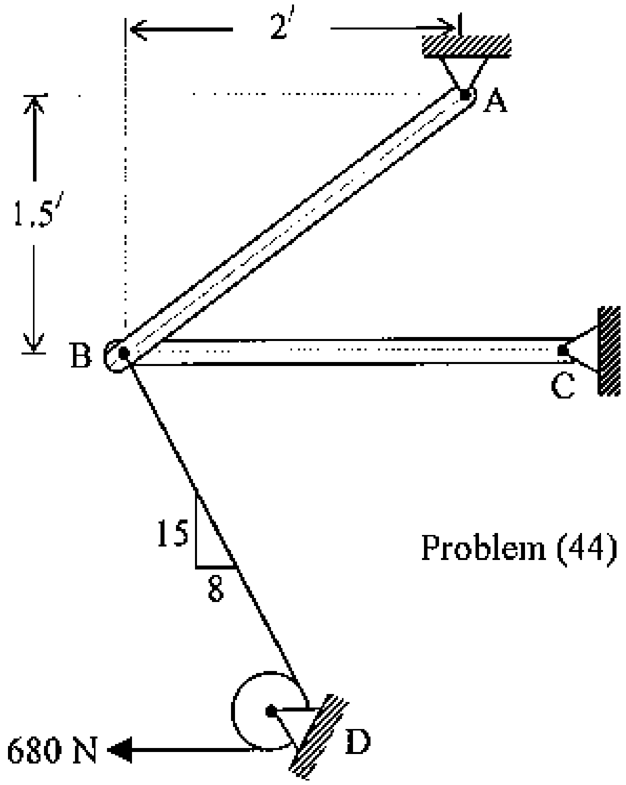
Problem (43): A rope is sluing around a homogeneous box of 200 lb. weight, in order to lift it by means of a crane. Neglecting friction between steel ring and rope, determine the force in the rope and angle α.

Answer (43): Force in the rope = 200 lb. (tension) & angle (α) = 120^0.

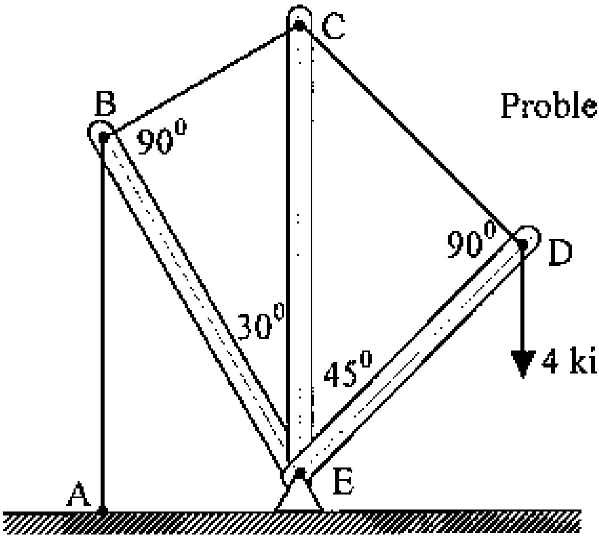


Problem (43)

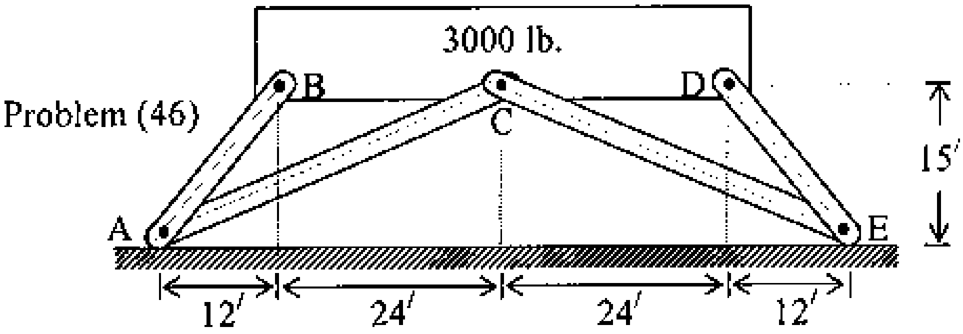
Problem (44): Determine the force in the member AB & BC.
Answer (44): F_{AB} = 1000 N (tension); F_{BC} = 1120 N (compression)



Problem (45): Determine tension in the cables AB, BC & CD.
Answer (45): T_{AB} = 4.618802154 lb.; T_{BC} = 2.309401077 lb.; T_{CD} = 2.828427125 lb.

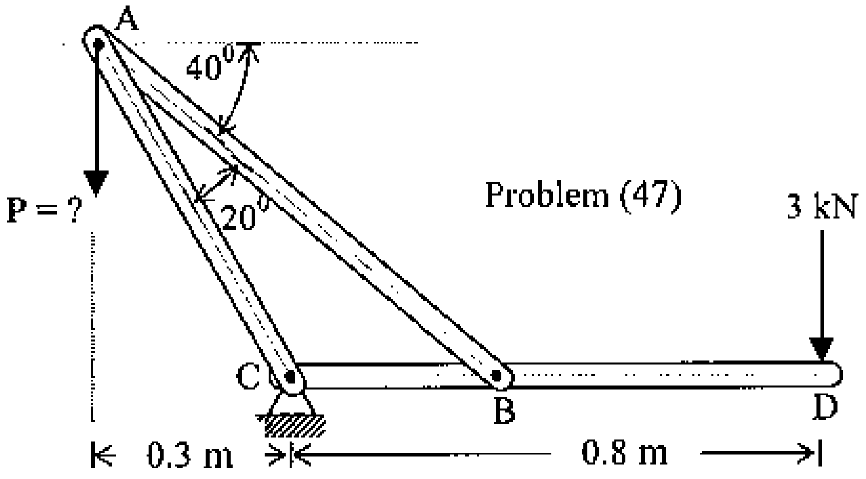


Problem (46): Determine the forces in members AB, AC, CE & DE. Contact surface at A & E is smooth.
Answer (46): F_{AB} = F_{DE} = 780.8688094 lb. (compression); F_{AC} = F_{CE} = 528.4552846 lb. (tension)



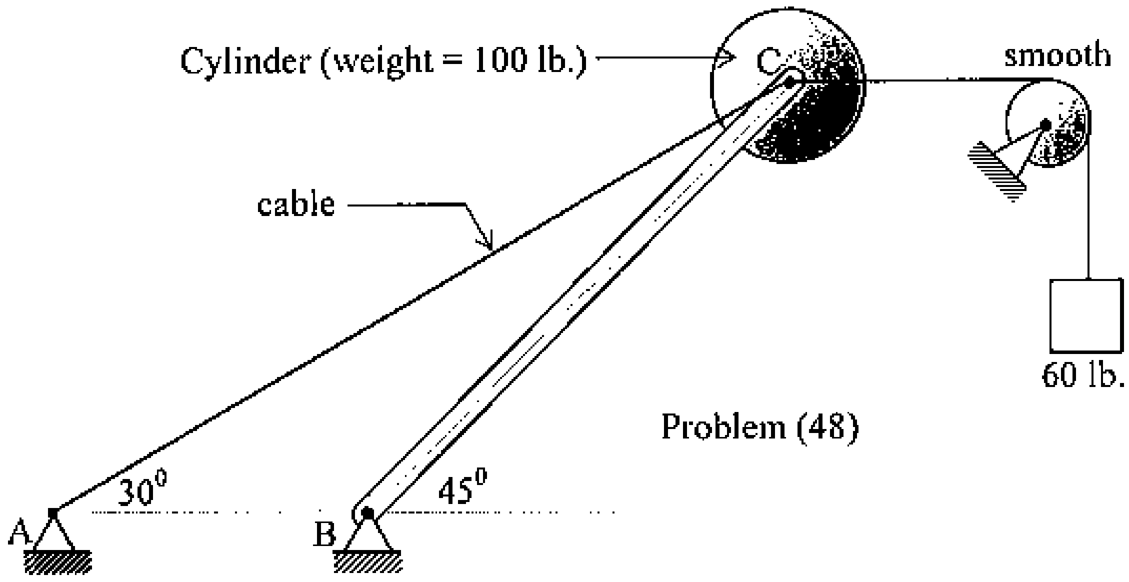
Problem (47): Determine force P to make the frame in equilibrium. Also determine the force in members AB & AC.

Answer (47): P = 8 kN (↓); F_{AB} = 11.6952176 kN (tension); F_{AC} = 17.91811291 kN (compression)

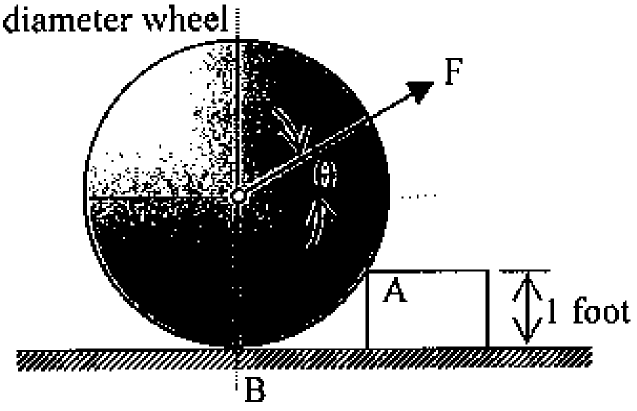


Problem (48): Determine the force in cable AC & member BC.

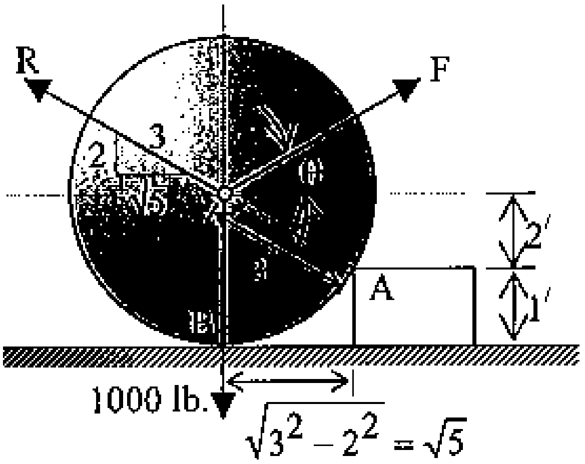
Answer (48): F_{AC} = 437.1281292 lb. (tension); F_{BC} = 450.5176206 lb. (compression)



Problem (49): A 1000 lb. wheel with a radius of 3 feet is acted upon by a force F, which tends to pull the wheel over the obstruction at A. At the instant the wheel is about to move, the reaction between the wheel and the ground is zero. What is the magnitude & sense of least force F at this instant?



Problem (49)



Solution (49)

Solution (49): $\sum F_x = 0\ (\rightarrow +\text{ve});\ \Rightarrow +F\cos\theta - R\times\frac{\sqrt{5}}{3} = 0$

$\sum F_y = 0\ (\uparrow +\text{ve});\ +F\sin\theta + R\times\frac{2}{3} - 1000 = 0$

Eliminating R from the above two equations

$\Rightarrow +2F\cos\theta + \sqrt{5}F\sin\theta = 1000\sqrt{5}$

$$\Rightarrow F = \frac{1000\sqrt{5}}{2\cos\theta + \sqrt{5}\sin\theta}$$

$$\Rightarrow F = \frac{1000\sqrt{5}}{\sqrt{(2)^2+(\sqrt{5})^2}\left[\frac{2}{\sqrt{(2)^2+(\sqrt{5})^2}}\cos\theta + \frac{\sqrt{5}}{\sqrt{(2)^2+(\sqrt{5})^2}}\sin\theta\right]}$$

$$\Rightarrow F = \frac{1000\sqrt{5}}{3\left[\frac{2}{3}\cos\theta + \frac{\sqrt{5}}{3}\sin\theta\right]}$$

let, $\sin\alpha = \frac{2}{3}$ & $\cos\alpha = \frac{\sqrt{5}}{3}$; $\alpha = \sin^{-1}\left(\frac{2}{3}\right) = \cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = 41.81031489^0$.

$$\Rightarrow F = \frac{1000\sqrt{5}}{3[\sin\alpha\ \cos\theta + \cos\alpha\ \sin\theta]} = \frac{\frac{1000\sqrt{5}}{3}}{\sin(\alpha+\theta)}$$

F will be minimum, if the quantity in the denominator (i.e. sin (α+θ)) is maximum. Maximum value of sin (α+θ) is 1 & $F_{\text{minimum}} = \frac{\frac{1000\sqrt{5}}{3}}{\sin(\alpha+\theta)} = \frac{\frac{1000\sqrt{5}}{3}}{1} = 745.3559925$ lb.

$$\Rightarrow \sin(\alpha+\theta) = 1 = \sin 90^0;\ \alpha+\theta = 90^0$$
$$\Rightarrow \theta = 90^0 - \alpha = 90^0 - 41.81031489^0 = 48.18968511^0.$$

Alternatively, for least value of F; $\frac{dF}{d\theta} = 0$.*

$$\Rightarrow F = \frac{1000\sqrt{5}}{2\cos\theta + \sqrt{5}\sin\theta}$$
$$\Rightarrow \frac{dF}{d\theta} = \frac{-1000\sqrt{5}}{(2\cos\theta + \sqrt{5}\sin\theta)^2}(-2\sin\theta + \sqrt{5}\cos\theta) = 0$$
$$\Rightarrow (-2\sin\theta + \sqrt{5}\cos\theta) = 0$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) = 48.18968511^0.$$

Substituting this value of $\theta = 48.18968511^0$ in the expression of F.

$$\Rightarrow F = \frac{1000\sqrt{5}}{2\cos\theta + \sqrt{5}\sin\theta} = \frac{1000\sqrt{5}}{2\cos 48.18968511^0 + \sqrt{5}\sin 48.18968511^0}$$
$$\Rightarrow F = 745.3559925 \text{ lb.}$$

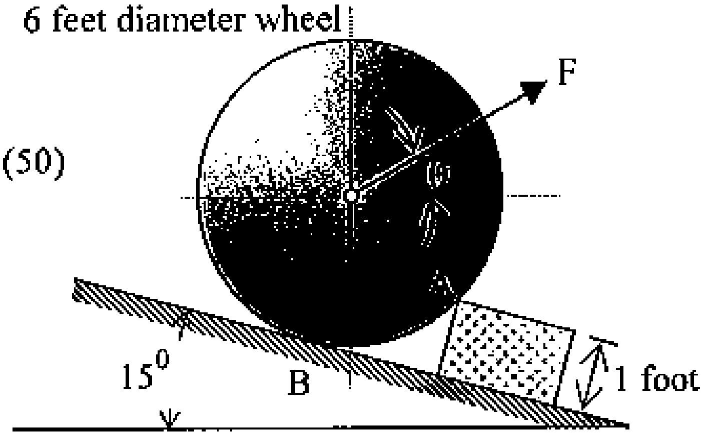
*for details, see any book on differential calculus (topics: maxima & minima).

Answer (49): F = 745.3559925 lb. & $\theta = 48.18968511^0$.

Problem (50): The wheel (weight 1000 lb. & diameter 6 feet) is on the point of rolling over the block (that is, the reaction at B is zero, but the wheel has not moved). What is the magnitude and sense of the least force F that will produce this condition?

Answer (50): F = 547.4125729 lb. & $\theta = 33.18968511^0$.

Problem (50)



Chapter Four

Moment & Parallel Coplanar Forces

How We Produce Turning or Swing Movements: Whenever a body is restrained by means of a hook, nail or hinge, and a force is applied to it, the only kind of motion of which it is capable is that of turning. Thus a door can only turn about hinges; the suspended nail swings at the end of the string to which it is attached; children playing at see-saw cause the plank to swing about the fallen tree upon which it is placed, and so on.

We express this by saying that the force exerts a moment upon the body. The greater the power of a force to cause a body to rotate, the greater is its moment.

Upon what does this moment of a force, i.e., its power to cause turning, depend? The answer is that it depends (a) upon the size, or strength, of the force, of the force, and (b) upon the distance from the point, or line of restraint, about which the force acts.

The first of these causes is obvious. A force of 10 lbs. must have twice the turning power of a force of 5 lbs., and ten times the turning power of a force of 1 lb. and so on.

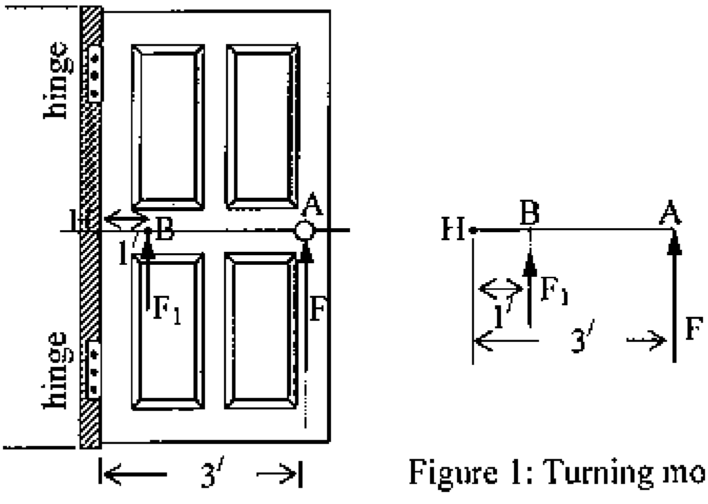


Figure 1: Turning moment on a door

The effect of the distance of the force from the point of restraint will be obvious if we think, for example, of the opening a door. The force F, which will just push the door open at the door-knob (figure 1) will not be sufficient to do so at B. A much stronger force F_1 is needed, and the nearer the force is to the hinge-line H the greater it must be if it is to succeed in opening the door.

Moment/Moment of a Force about a Point: The moment of a force is its tendency to rotate the body upon which it acts about some point. A moment is quantified by the product of the force and the perpendicular distance between the axis of rotation and the force's line of action. This distance is called moment arm & the axis of rotation is called moment center or fulcrum.

The moment of a force is a measure of its ability to cause a body to rotate about a point (called moment centre) or about an axis (called moment axis).

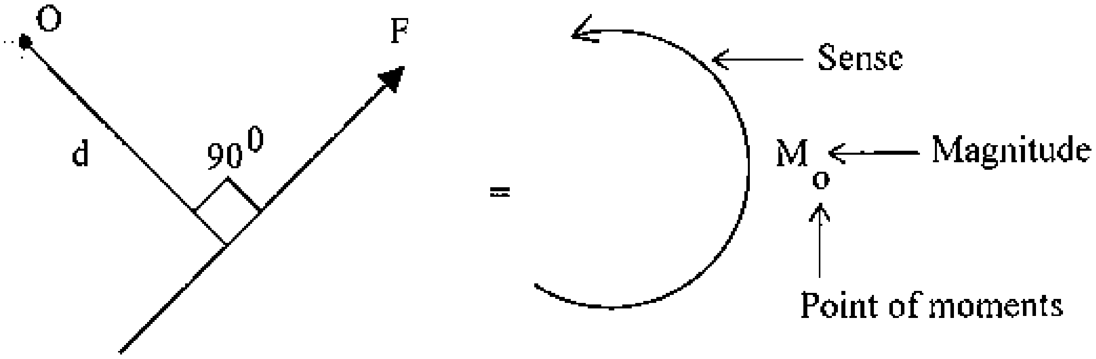


Figure 2: Explanation of moment

Measurement of a Moment: Two quantities are involved in expressing the values of a *moment* or *turning effect.*

(1) The magnitude of applied force.

(2) The perpendicular distance between the line of action of the force and the point about which turning being considered, i.e., the fulcrum. This distance is termed the *arm of the moment.*

The value of the moment increases directly as these two quantities increases, so that we multiply them together to obtain the actual moment.

$$M_O = F \times d$$

M_O = Moment of force F about point O

O = Moment center / axis of rotation

F = Magnitude of force

d = moment arm

It is important to realize that the moment of a force about an axis passing through its line of action is zero (because the moment arm is zero).

A Simple Experiment on Moments: Take a meter scale and drill a neat hole exactly in the middle so that it balances when suspended to swing easily on a brass peg fitted into a beam or rod to act as a fulcrum (figure 3).

Take two weights P and Q, say 50 grams and 100 grams respectively, and by means of loops of thread slip them over the suspended scale, one on either side of the fulcrum C. start with the 50 gm. Weight near one end, and slide the other weight to fine the exact position to produce to produce the balance or equilibrium of the two weights. It will be found that, whatever the positions of the 50 gm. weight, the 100 gm. weight must always be moved to half its distance from the fulcrum and on the opposite side of the scale to produce balance.

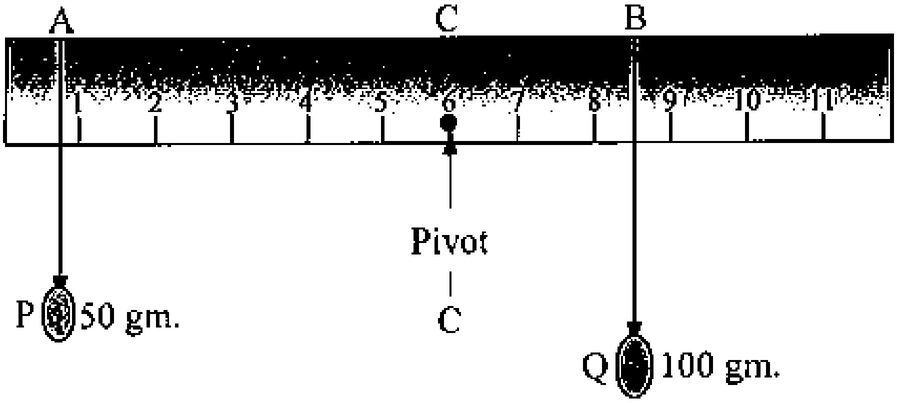


Figure 3: A Simple Experiment on Moments

Remove these two weights and repeat the experiment for any other pair of weights. The results will always show that, as in figure 3, in order to get balance, the moment of the force P about C in a counter-clockwise direction is exactly equal to the moment of force Q about C in a clockwise direction,

$$\text{i.e., } P \times AC = Q \times BC$$

The experiment proves we are right in measuring moments by the product of the force and the distance from the point of restraint, or fulcrum, about which it acts.

Methods of Finding Moment of a Force: based on the definition of the moment about a point and Varignon's* theorem, various methods of finding the magnitude of moments are possible. They are discussed with respect to two dimensional problem (see below).

(i) From the definition of moment and referring to moment figure 4(a)

$$\mathrm{M_o = F \times d}$$

$$\mathrm{M_o = F \times (d_1 + d_2)}$$

$$M_o = F \times (y\cos\theta + x\sin\theta)\text{, clockwise}$$

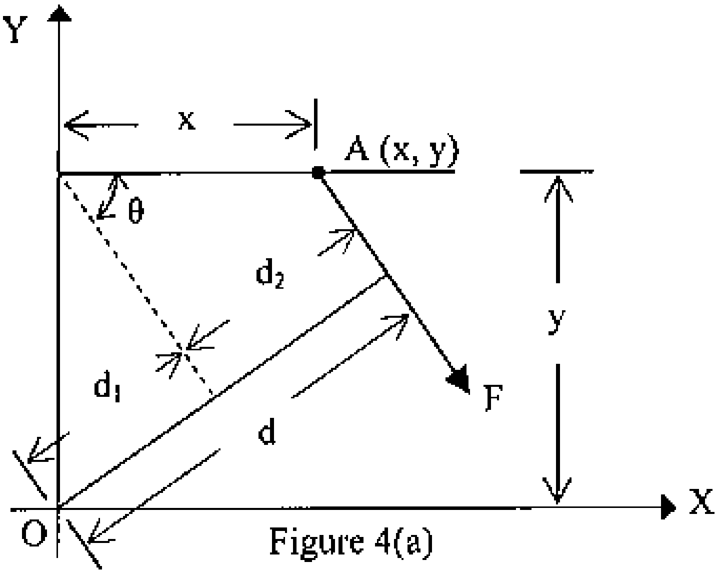


Figure 4(a)

(ii) Resolving the force into its components in the coordinate direction (see figure 4(b))

$$M_o = F_x y + F_y x$$

$$M_o = F\cos\theta \times (y) + F\sin\theta \times (x)$$

$$M_o = F \times (y\cos\theta + x\sin\theta)\text{, clockwise}$$

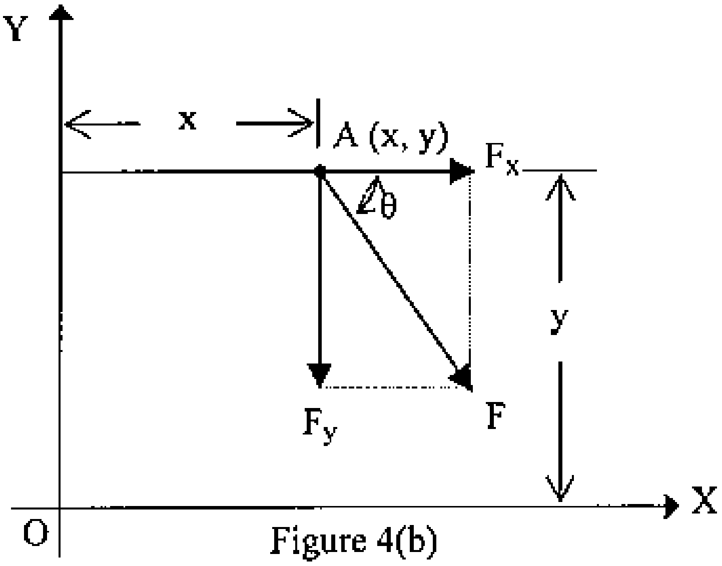


Figure 4(b)

(iii) Transmitting the force to the Y axis and then resolving it into its component (see figure 4(c))

$$M_o = F_x \times OA$$

$$M_o = F_x \times (y + x\tan\theta)$$

$$M_o = F\cos\theta \times (y + x\tan\theta)$$

$$M_o = F \times (y\cos\theta + x\sin\theta)\text{, clockwise}$$

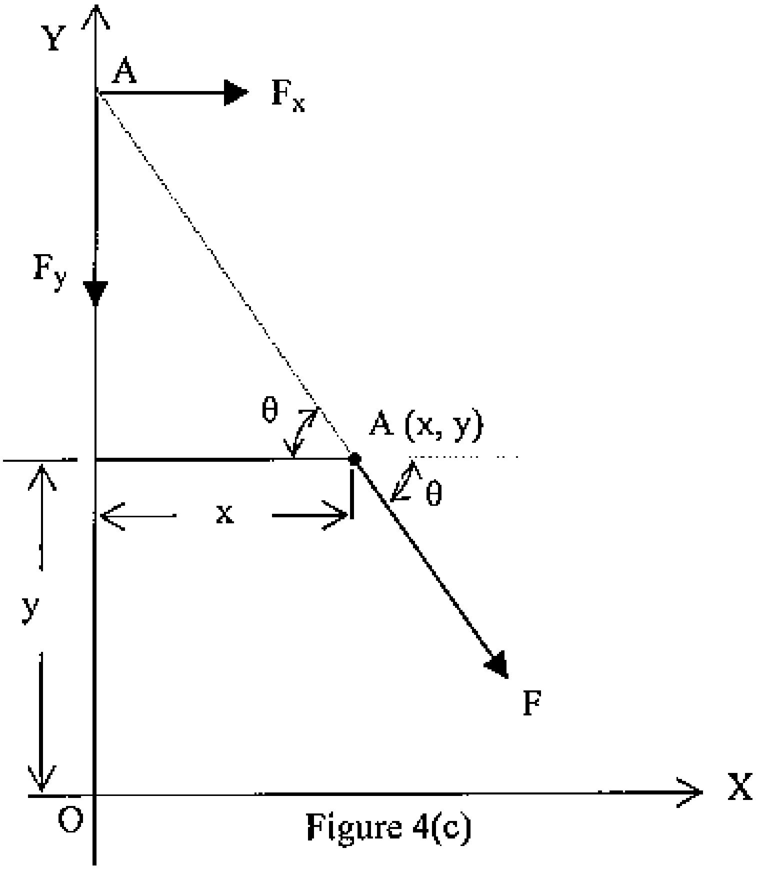


Figure 4(c)

(iv) Transmitting the force to the X axis and then resolving it into its components (see figure 4(d))

$$M_o = F_y \times OB$$

$$M_o = F_y \times (x + y\cot\theta)$$

$$M_o = F\sin\theta \times (x + y\cot\theta)$$

$$M_o = F \times (y\cos\theta + x\sin\theta)\text{, clockwise}$$

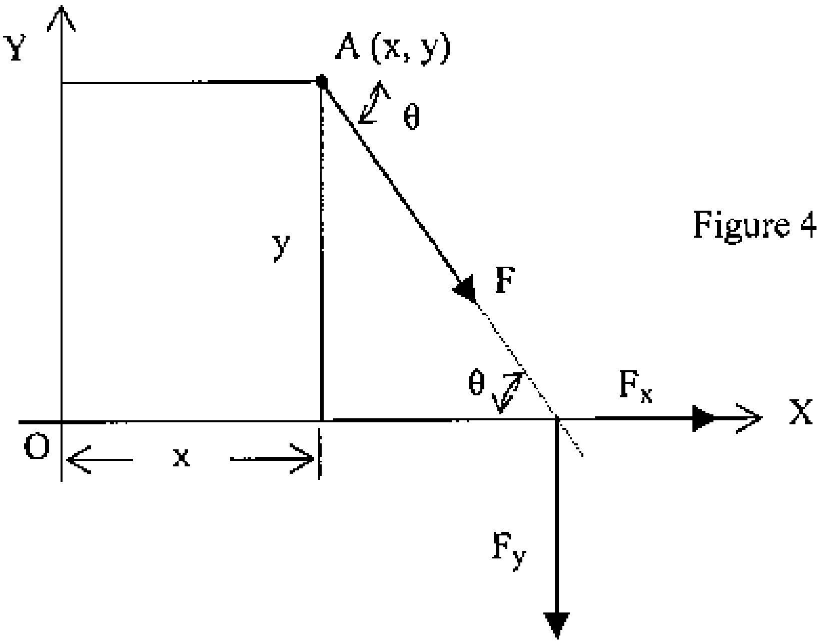


Figure 4(d)

(v) Moment may be found by vector operations

$$\underline{r}_{OA} = x\,\underline{i} + y\,\underline{j}$$

$$\underline{F} = F\,(l\,\underline{i} + m\,\underline{j})$$

$$l = \cos\theta \text{ and } m = -\cos(90^{o} - \theta) = -\sin\theta$$

$$\underline{F} = F\,(\cos\theta\,\underline{i} - \sin\theta\,\underline{j})$$

$$\underline{M}_O = \underline{r}_{OA} \times \underline{F} = (x\,\underline{i} + y\,\underline{j}) \times F\,(\cos\theta\,\underline{i} - \sin\theta\,\underline{j})$$

$$M_o = F \times (y\cos\theta + x\sin\theta), \text{ clockwise}$$

In two-dimensional problems, any of the above five techniques can be used to find the moment of a force about a point conveniently. But, in three-dimensional problems the vector approach is convenient.

[*Pierre Varignon (1654-1722), a Frenchman, and a contemporary of Newton, although not generally counted among the giants of science, made an extremely important contribution to mechanics in his principle of moments. He is also credited by Girwin in his Historical Appraisal of Mechanics as having been the first to evolve the differential equations of motion.]

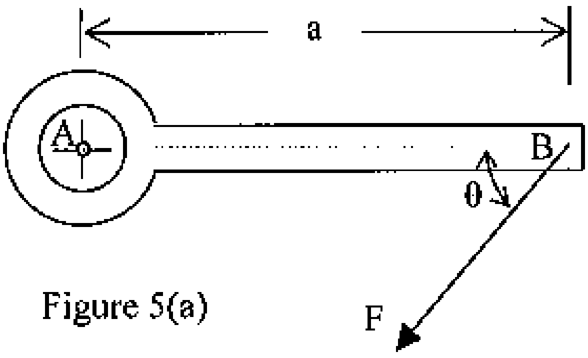


Figure 5(a)

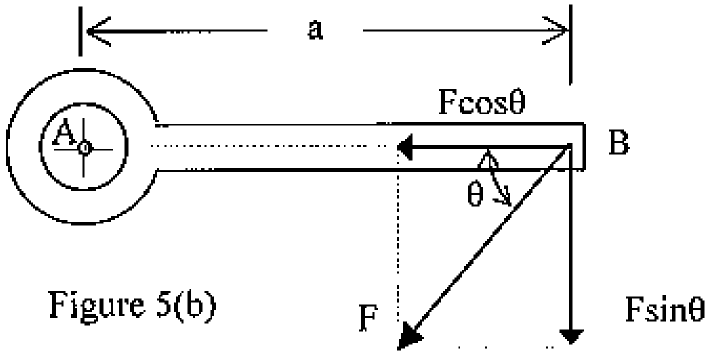


Figure 5(b)

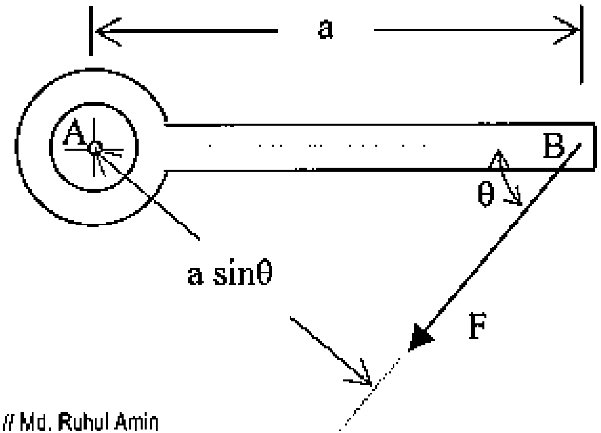


Figure 5(c)

Example: Determine moment of force F about point A (see figure 5(a)).
Method (i): Resolving the force F into two components along AB & perpendicular to AB (see figure 5(b)).
Component of F along AB = $F\cos\theta$
Component of F along AB = $F\sin$
Moment of force of about A, $M_A = (F\cos\theta) \times (0) + (F\sin\theta) \times (a)$
$= Fa\sin\theta$ (clockwise)
Method (ii): Perpendicular distance of line of action of F from point A = $a\sin\theta$ (see figure 5(c)).
Moment of force of about A, $M_A = (F) \times (a\sin\theta)$
$= Fa\sin\theta$ (clockwise)

Units of Moment: The units of a moment are the product of the units of force and distance. Example: N-m (Newton-meter), kN-m (kiloNewton-meter), in-lb (inch-pound), ft-lb (foot-pound), kip-ft (kilopound-foot), kip-in (kilopound-inch).

Types of Moments: Moments are of the following two types
(1) Clockwise moments
(2) Anticlockwise / Counterclockwise moment

Clockwise moment: It is the moment of a force, whose effect is to turn or rotate the body, about the point in the same direction in which hands of a clock move as shown in figure 6.

Anticlockwise / Counterclockwise moment: It is the moment of a force, whose effect is to turn or rotate the body, about the point in the opposite direction in which hands of a clock move as shown in figure 6.

Sign of Moment: No definite rule. Conventionally, Clockwise moment is taken as positive & anticlockwise moment as negative.

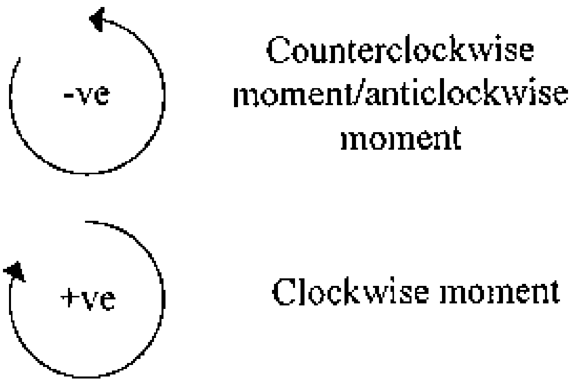


Figure 6(a): Types of moment

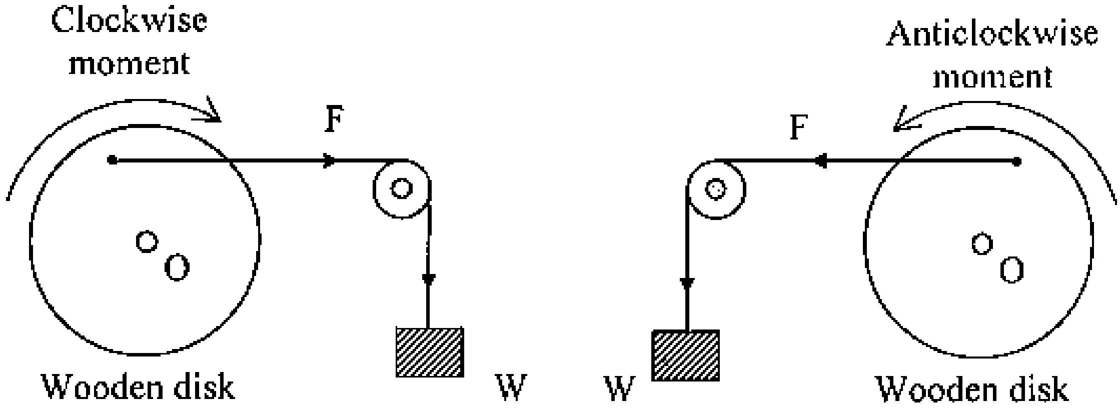


Figure 6(b): Types of moments

Varignon's Theorem /Varignon's Principle of Moments /Law of Moments: The French Mathematician Pierre Varignon (1654-1722) gave the following theorem which is also known as the principles of moments.

"The algebraic sum of moments of a system of forces about a point (this point is called moment centre) is equal to the moment of their resultant force about the same point (i.e. moment centre)."

Actually, Varignon stated it in the context of coplanar forces and proved it. However it can be generalized to two or three dimensional problems. Geometric proof & Vector proof of the theorem is quite simple.

Parallel Forces: The forces, whose lines of action are parallel to each other, are called parallel forces.

Classification of Parallel Forces: The parallel forces may be broadly classified into the following two categories, depending upon their direction

(1) Like parallel forces
(2) Unlike parallel forces

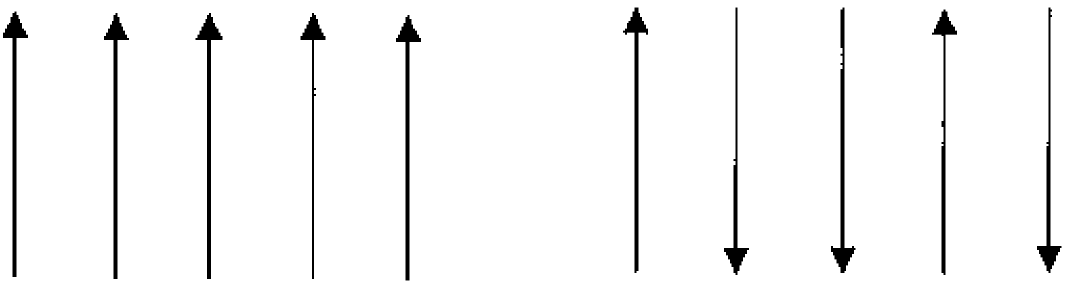


Figure 7(a): Like parallel forces

Figure 7(a): Unlike parallel forces

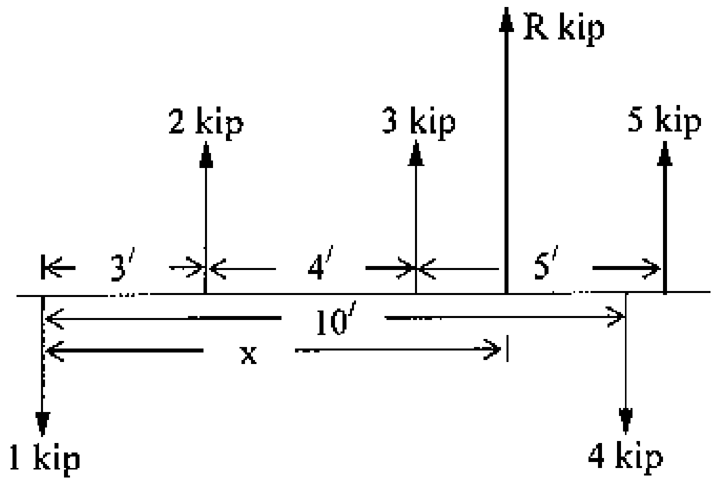
Figure 7: Parallel forces

The forces, whose line of actions are parallel to each other and all of them act in the same direction as shown in figure 7(a) are known as like parallel forces.

The forces, whose line of actions are parallel to each other and all of them do not act in the same direction as shown in figure 7(b) are known as unlike parallel forces.

Experiment on Parallel Forces: See page no 71 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St Paul's House Warwick Lane, London EC4.

Find the resultant of the following parallel force system



Magnitude of resultant (R) = algebraic sum of forces = (2 + 3 + 5 – 1 – 4) kip = 5 kip.
Direction: Vertically upwards.
Position: Let 'x' ft = distance from the left (on the 1 kip force) measured towards the right.
(R kip × x feet) = (2 kip × 3 feet) + (3 kip × 7 feet) + (5 kip × 12 feet) – (1 kip × 0 feet) – (4 kip × 10 feet)
⇒ (5 kip × x feet) = (2 kip × 3 feet) + (3 kip × 7 feet) + (5 kip × 12 feet) – (1 kip × 0 feet) – (4 kip × 10 feet)

$$\Rightarrow x = \frac{2\times3+3\times7+5\times12-1\times0-4\times10}{5} = 9.40 \text{ feet.}$$

Couples: If, in such an example as the one solved above, the sum of the forces acting upwards equaled the sum of those acting downwards, two possibilities would ensue:

(i) If the upward and downward resultant forces acted in the same vertical line, the system would be in equilibrium.

(ii) If they acted out of line the net effect of the system would be one of rotation. A system of forces, such as that suggested in (ii), is known as a 'couple'.

Definition: A couple consists of two equal, unlike parallel forces acting out of line. The effect of a couple on a body in one of pure rotation.

Couple-moment: Two parallel forces that are equal in magnitude but opposite in direction and separated by a definite distance are said to form a couple.

The rotational effect of the couple is called the couple moment. The distance between the parallel forces forming a couple is called the arm of the couple.

Moment of couple = Force × arm of couple.

A couple has a 'clockwise' or an 'anticlockwise' moment according to the manner in which it tends to cause rotation.

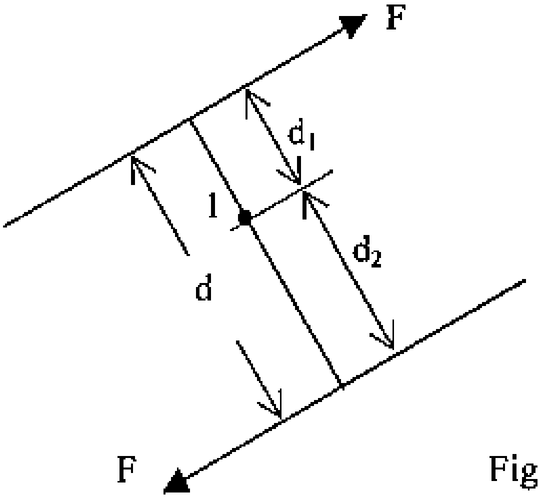


Figure 8(a)

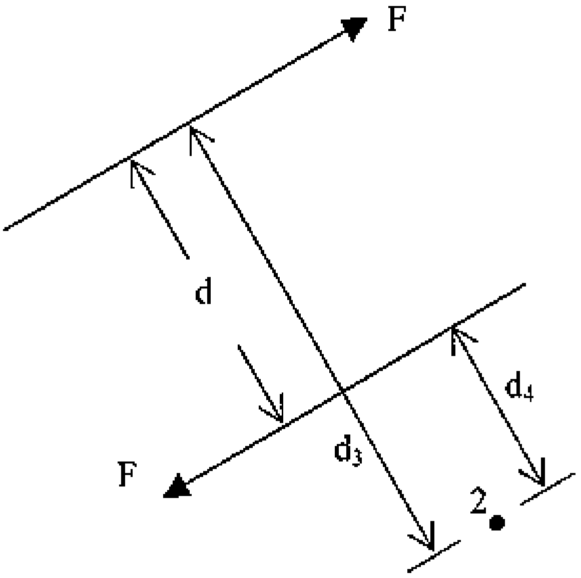


Figure 8(b)

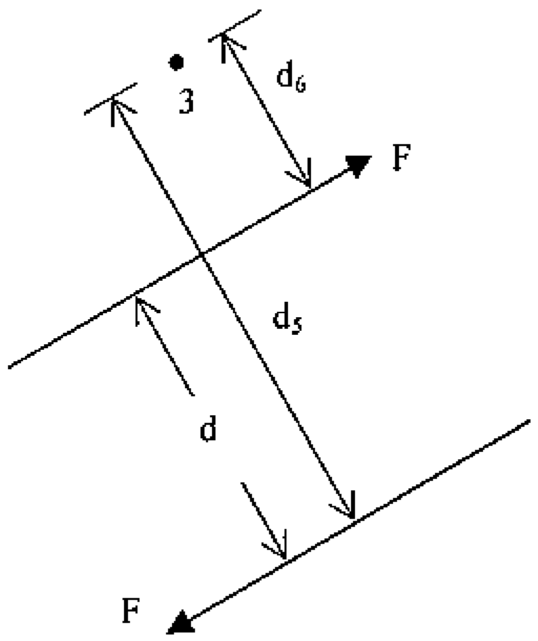


Figure 8(c)

Let the magnitude of the forces forming the couple be F and the perpendicular distance between the two forces be d (figure 8(a)). Consider the magnitude of the moment of these forces about point 1. Let its value be M_1. Then,

$$M_1 = Fd_1 + Fd_2$$

$$M_1 = F(d_1 + d_2) = Fd \text{ (clockwise)}$$

Now consider the moment of the forces about point 2 which is outside the two forces as shown in figure 8(b). Let M_2 be the moment. Then

$$M_2 = Fd_3 - Fd_4$$

$$M_2 = F(d_3 - d_4) = Fd \text{ (clockwise)}$$

Similarly, even in the case shown in figure 8(c) the moment is Fd only. Thus the couple moment about any point is a constant. Since the two forces are equal and opposite, their resultant force is zero, hence there is no translating effect.

Now we can list the following characteristics of a couple:

(1) A couple consists of pair of equal and opposite parallel forces which are separated by a distance.
(2) The translatory effect of a couple on the body is zero.
(3) The rotational effect (moment) of a couple about any point in a constant and it is equal to the product of the magnitude of the force and the perpendicular distance between the two forces.
(4) The direction of the couple moment is at right angles to the plane of the couple.
(5) Two couples will be in equilibrium if their moments are equal in magnitude but opposite in direction, i.e., if one is a clockwise couple, the other must be a counter-clockwise couple, and so they must cancel each other out and the resulting turning effect will be nil (i.e. two couples in the same plane of equal & opposite moments, balance).

(6) Any two coplanar couples of equal moments in the same sense, are equivalent to one another.

(7) A number of coplanar couples compound into a simple couple whose moment is equal to the algebraic sum of the moments of the various couples.

(8) A couple & a force in the same plane cannot be in equilibrium.

[* the name couple was first used by Louis Poinsot (1777-1959) & his theory of couples was presented in *Éléments de Statique* (1806). Poinset was engineer of roads & bridges and in 1804 was made professor of mathematics at the Lyceum Bonaparte.]

Resolution of a Force into a Force and a Couple: It many engineering mechanics problems, it will be advantageous to resolve a force at a point on a body into a force acting at some other suitable point on the body and a couple. In figure 9(a), F is a force acting on a body at A.

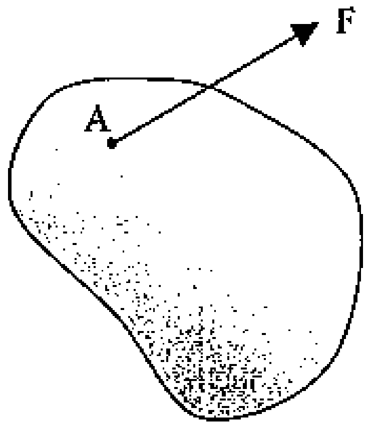


Figure 9(a)

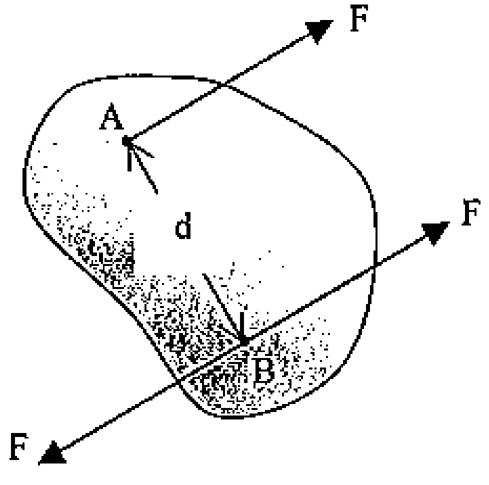


Figure 9(b)

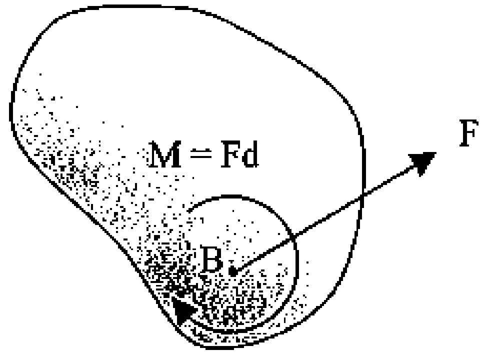


Figure 9(c)

By applying equal and opposite forces F at B on the body, the system of forces is not disturbed. Hence the system of forces in figure 9(b) is the same as the system given in figure 9(a). Now the original force F at A and the opposite force F at B form a couple. Its effect may be replaced by the couple moment M = Fd. Hence, the system in figure 9(b) can be replaced by the system shown in figure 9(c). thus, the given force F at A is replaced by a force F at B and a moment M = Fd.

Alteration of the Arm of a Couple-Equivalent Couples: We have seen that the moment of a couple is equal to the product of one of the forces and the arm. As long as this product is constant, both arm and force may be altered. This in figure 10, the couple whose moment = P × p can have a smaller arm by q increasing the force to Q such that Q × q = P × p, giving $Q = \frac{P \times p}{q}$.

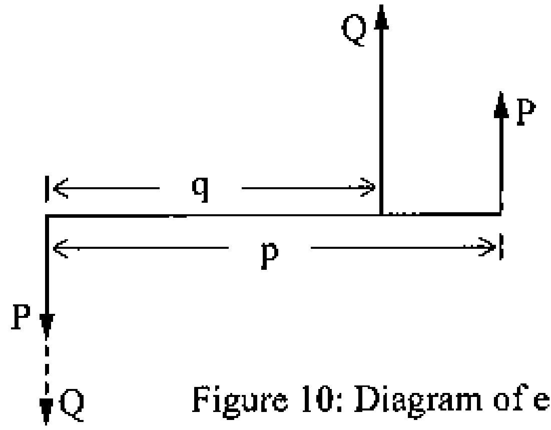


Figure 10: Diagram of equivalent couples

Example of a Couple – The Capstan: A capstan is a well-known device at dock-sides, abroad ships and elsewhere. Its purpose is to move or operate heavy bodies attached to them by means of a rope R (figure 11) wound around the body of the capstan. A long rod projects on either side from prepared holes in the rotating cap C. two men work with the mechanism at A and B equidistant from C, applying a couple P–P in the directions shown. The moment of the couple so produced rotates the capstan. It will be seen that since the lever arm of the rope about the center is very small compared with the much greater arm of the rotating couple, a very large pull is obtained in the rope sufficient to deal with the heavy object it is intended to move.

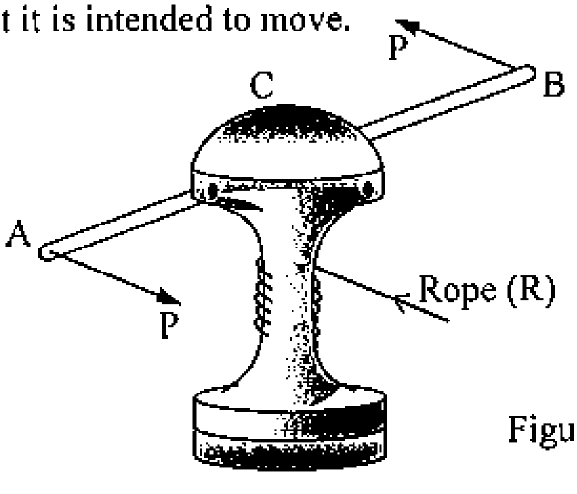
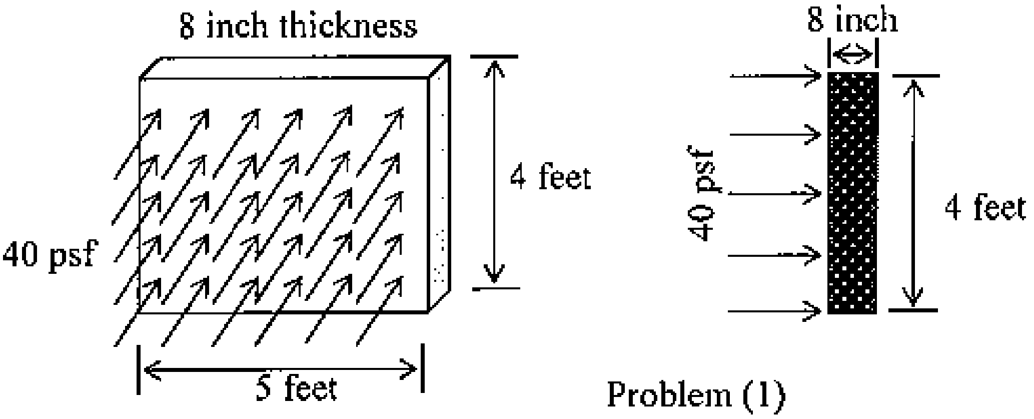


Figure 11: Capstan

Experimental Verification of Principle of Lever: See page no 52 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St. Paul's House Warwick Lane, London EC4.

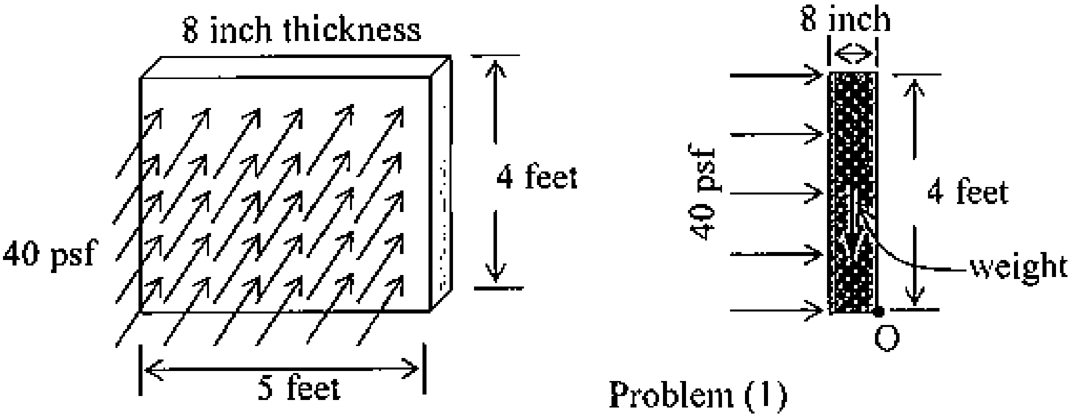
Problems

Problem (1): A concrete wall of density 150 lb/ft^3 ($pcf = lb/ft^3$), measures 5 ft × 4 ft × 8 inch. There is a horizontal pressure 40 psf ($psf = lb/ft^2$) on one side (due to wind). Will the block overturn?



Problem (1)

Solution (1): The block is tend to overturn about point O. The 40 psf lateral force try to overturn & vertical weight of block try to prevent it from overturning.



Problem (1)

Overturning moment about point O, $\sum M_O$ = (40 psf * 5 ft * 4 ft) * (2 ft) = 1600 ft-lb.

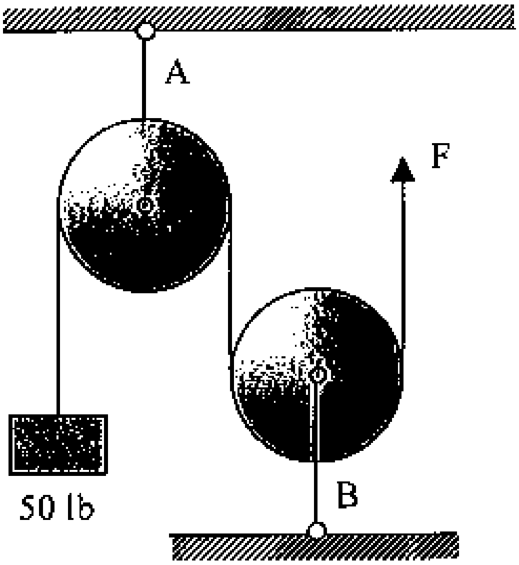
Restoring moment about point O, $\sum M_O$ = (150 pcf * 5 ft * 4 ft * 8/12 ft) * (4/12 ft) = 666.6666667 ft-lb.

Since, overturning moment > restoring moment; block will overturn about point O.

Answer (1): The block will overturn.

Problem (2): What force F is needed to raise the 50 lb weight and what are the tensions in the cables A & B?

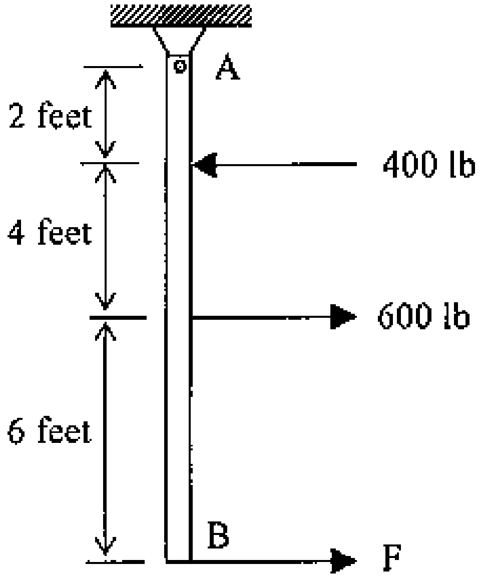
Answer (2): F = 50 lb; tension in cable A = 100 lb & tension in cable B = 100 lb.



Problem (2)

Problem (3): The vertical weightless bar, supported at A, is subjected to the two horizontal forces. What horizontal force F is needed for equilibrium? What is the pin reaction at A?

Answer (3): F = 333.333 lb (←); R_{AX} = 133.333 lb (→); R_{AY} = 0 lb.



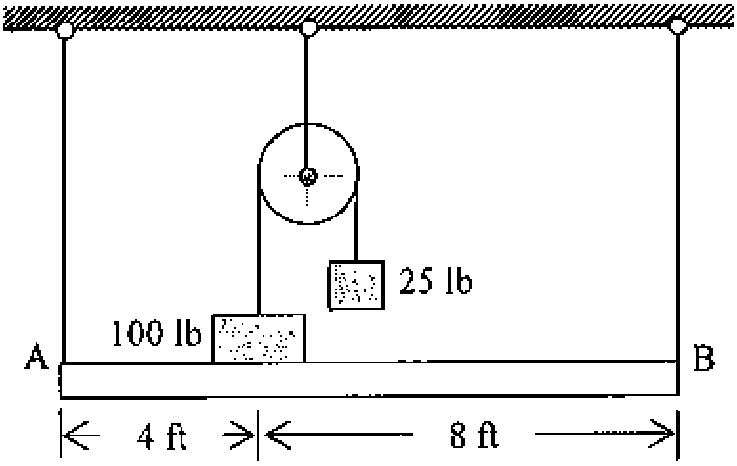
Problem (3)

Problem (4): Rework problem (2) considering the bar has a weight of 50 lb.

Answer (4): F = 333.333 lb (←); R_{AX} = 133.333 lb (→); R_{AY} = 50 lb (↑)

Problem (5): The 40 lb platform is supported by two vertical ropes at A & B. The 25 lb weight is supported by a frictionless 8 lb pulley. What is the tension in each rope?

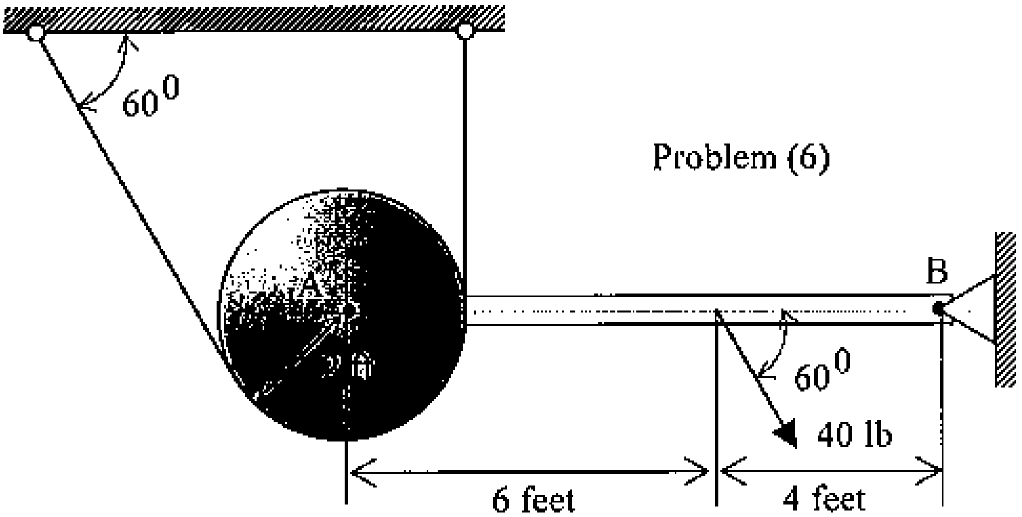
Answer (5): T_A = 70 lb (=35 lb. in each rope); T_B = 45 lb (=22.5 lb. in each rope)



Problem (5)

Problem (6): The frictionless pulley weighs 20 lb. The weightless horizontal beam also supports a 40 lb load. What is the tension around the rope passes around the pulley & pin reactions at B?

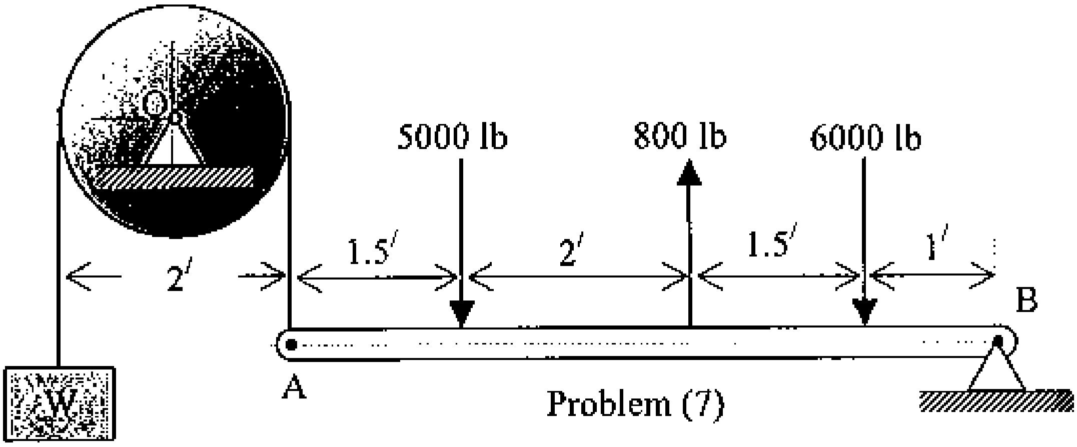
Answer (6): Tension = 18.1436 lb; B_X = 10.9282 lb (←); B_Y = 20.7848 lb (↑)



Problem (6)

Problem (7): Determine the weight W and the reaction at B if the link AB is in equilibrium. There is no friction at the pulley.

Answer (7): W = 4416.6667 lb (↓); R_{BX} = 0 lb & R_{BY} = 5783.3333333 lb (↑)



Problem (7)

Solution (7): From the free-body diagram of link AB

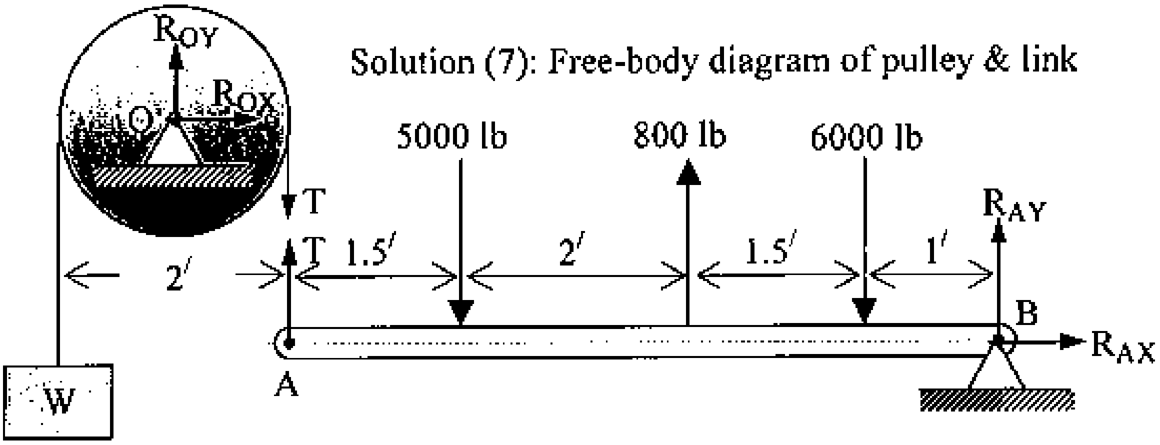
$\sum F_X = 0$ (→+ve); $R_{AX} = 0$

$\sum M_B = 0$ (clockwise +ve); $+(T\times 6) - (5000\times 4.5) + (800\times 2.5) - (6000\times 1) = 0$

$\Rightarrow T = 4416.666667$ lb. (tension)

$\sum F_Y = 0$ (↑+ve); $+R_{AY} - 6000 + 800 - 5000 + T = 0$

$\Rightarrow R_{AY} = 6000 - 800 + 5000 - T = 6000 - 800 + 5000 - 4416.666667 = +5783.333333$ lb. (↑)



Solution (7): Free-body diagram of pulley & link

From the free-body diagram of pulley

$\sum F_X = 0$ (→+ve); $R_{OX} = 0$

$\sum M_O = 0$ (clockwise +ve); $+(T\times 1) - (W\times 1) = 0$; $\Rightarrow T = W$

$\Rightarrow W = 4416.666667$ lb. (↓)

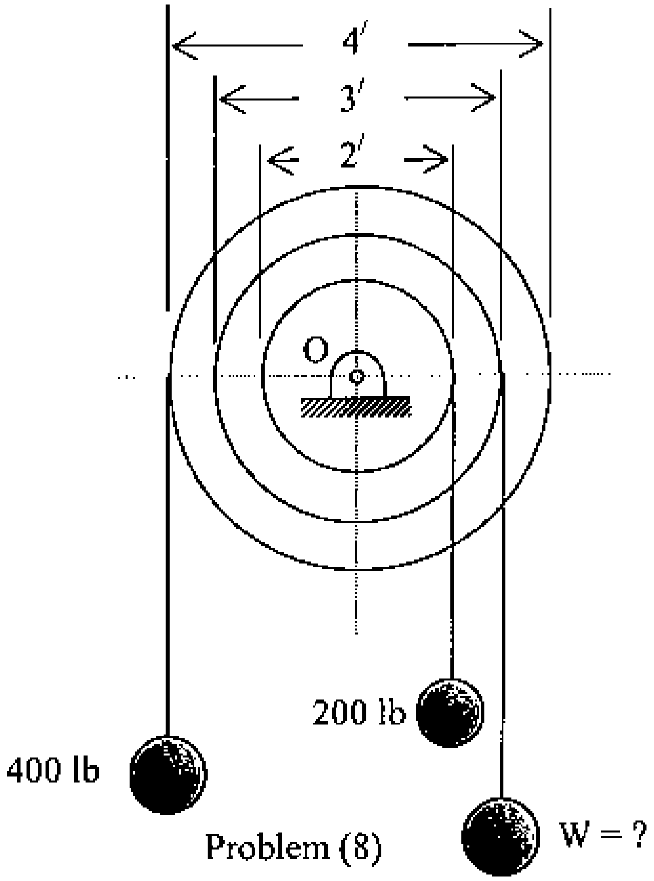
$\sum F_Y = 0$ (↑+ve); $+R_{OY} - W - T = 0$; $\Rightarrow R_{OY} = W + T = 8832.333333$ lb. (↑)

Problem (8): Three weights (400 lb, 200 lb, & W) are suspended in the same plane. What should be the value of W if $\sum M_O = 0$? What will be the reaction at O?

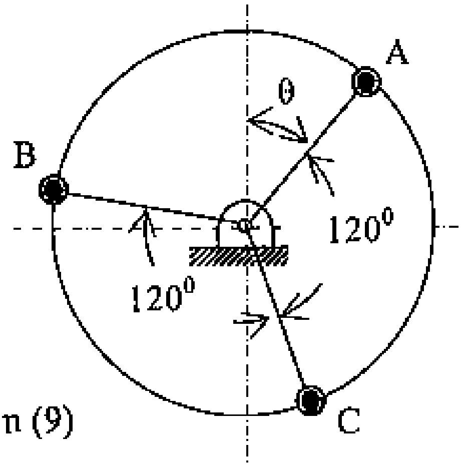
Answer (8): W = 400 lb (↓); $R_{OX} = 0$ lb; $R_{OY} = 1000$ lb (↑).

Problem (9): Three weights, A = 5 lb, B = 10 lb, and C = 15 lb, are attached to the rims of a uniform wheel at a 2 feet radius and spaced 120^0 apart. If the wheel is free to rotate about its axis, what angle θ will the radius point A make with the vertical when the wheel at rest (stable position)?

Answer (9): $\theta = 30^0$



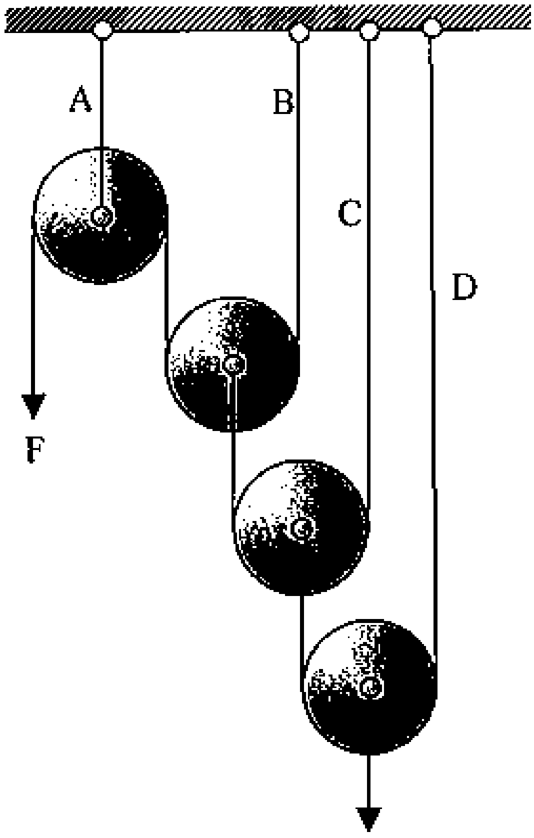
Problem (8)



Problem (9)

Problem (10): In the system of shaves, what force F will hold a weight of 800 pound in equilibrium? What will be the tension in cables A, B, C & D? There are no frictional losses at the axes.

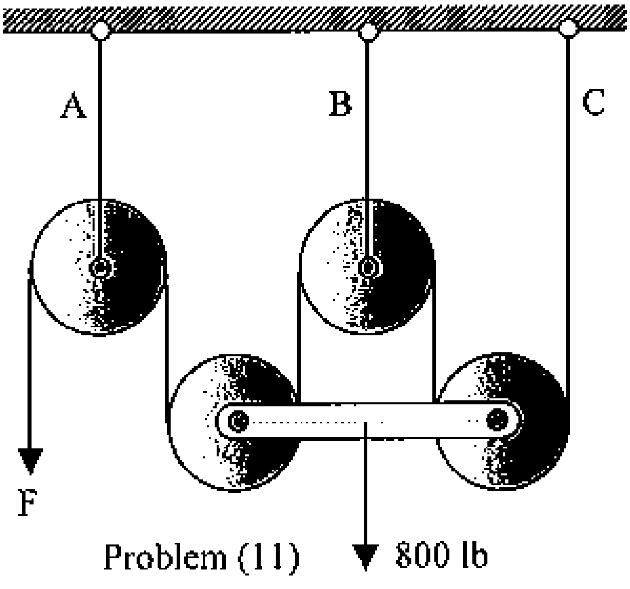
Answer (10): F = 100 lb., T_A = 200 lb., T_B = 100 lb., T_C = 200 lb. & T_D = 400 lb.



Problem (10) 800 lb

Problem (11): In the system of shaves, what force F will hold a weight of 800 pound in equilibrium? What will be the tension in cables A, B & C? There are no frictional losses at the axes.

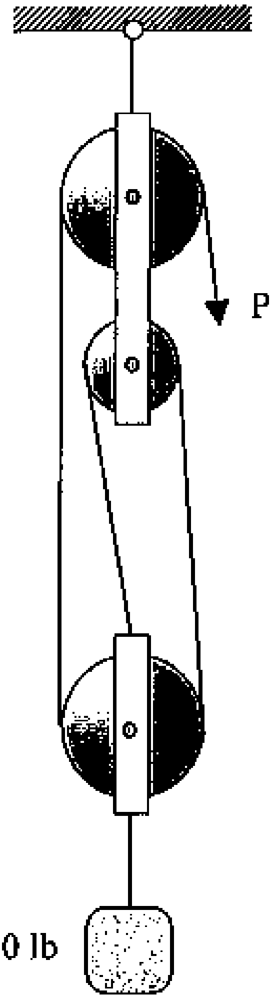
Answer (11): F = 200 lb., T_A = 400 lb., T_B = 400 lb. & T_C = 200 lb.



Problem (11) 800 lb

Problem (12): What force P is needed to hold the 1000 pound weight in equilibrium? Assume ropes are parallel.

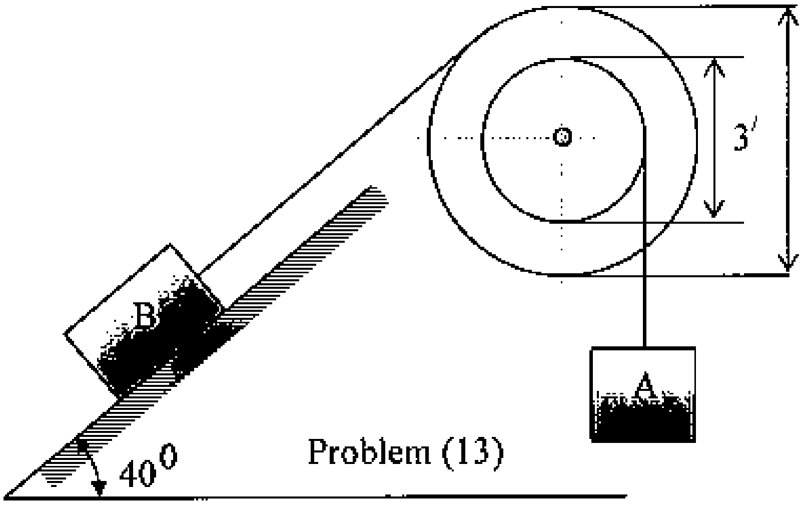
Answer (12): P = 333.333 pound.



Problem (12)

Problem (13): An ore car B which weighs 20 tons is balanced by a weight A. What should be the weight A if there is no friction at any point? Find also the plane reaction & tension in the chord.

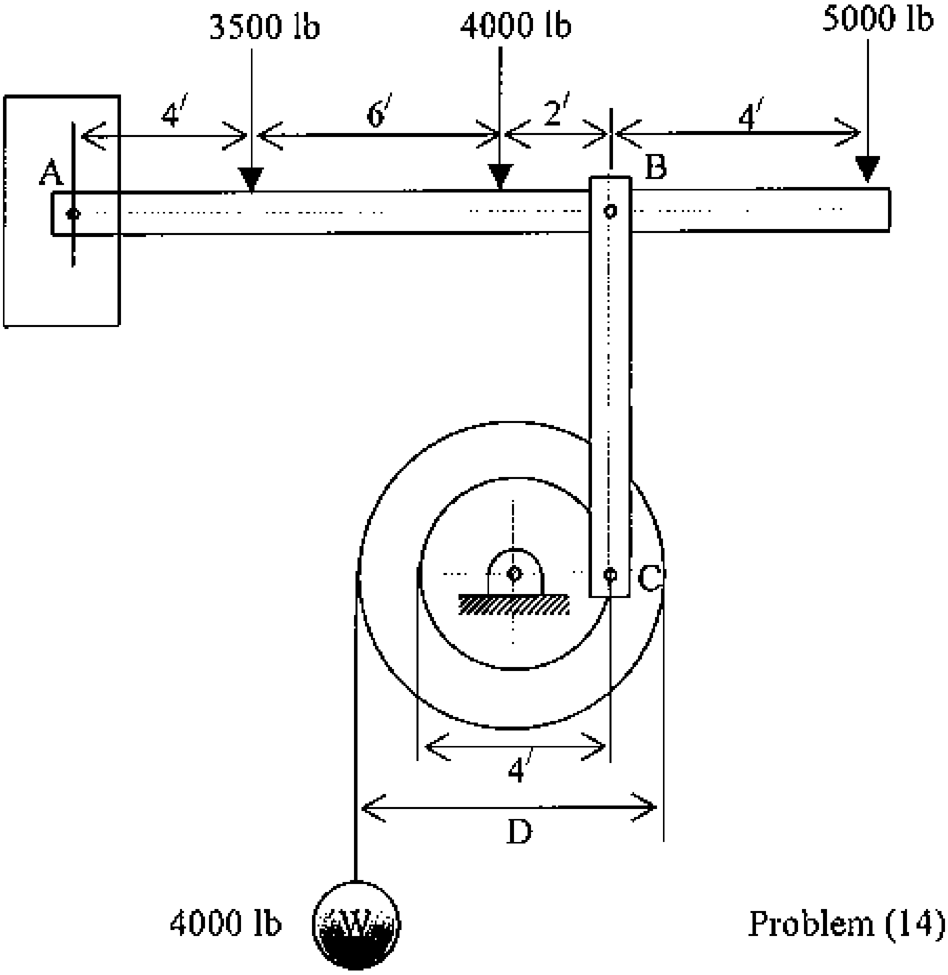
Answer (13): $W_A = 21.4263$ tons; $R_B = 15.321$ tons; T = 12.855 tons.



Problem (13)

Problem (14): The beam AB is loaded as shown in figure. Equilibrium is maintained by the weight W = 4000 lb. suspended from a pulley. What should be the diameter of the pulley? Find also the pin reaction at A and force in bar BC.

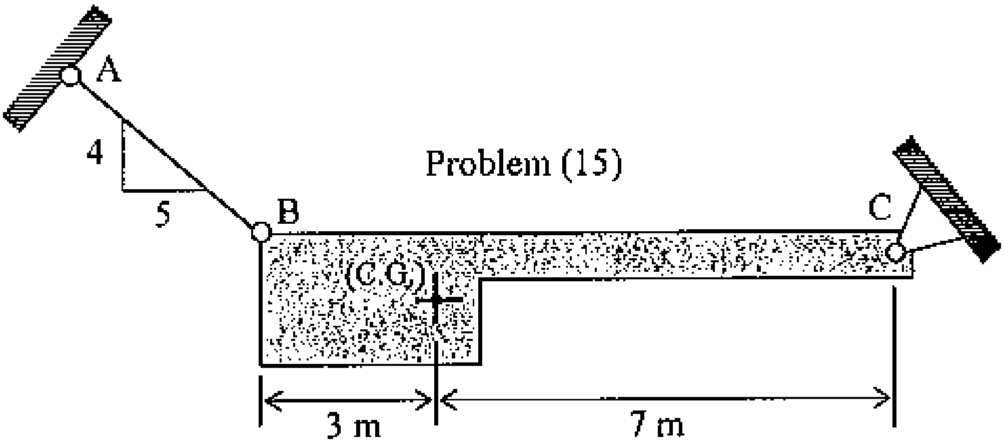
Answer (14): Diameter of pulley (D) = 12 feet, $R_{AX} = 0$ lb., $R_{AY} = 500$ lb. (↓) and force in bar BC = 12000 lb (compression).



Problem (14)

Problem (15): Compute the supporting force at pin C and cable tension AB. The block has a mass of 100 kg having the centre of gravity as shown.

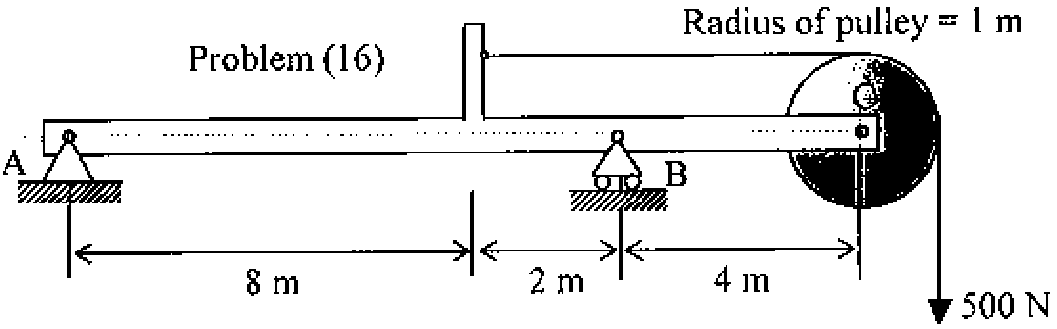
Answer (15): $R_{cx} = 87.50$ kg (→); $R_{cy} = 30$ kg (↑); $T_{AB} = 112.054$ kg



Problem (15)

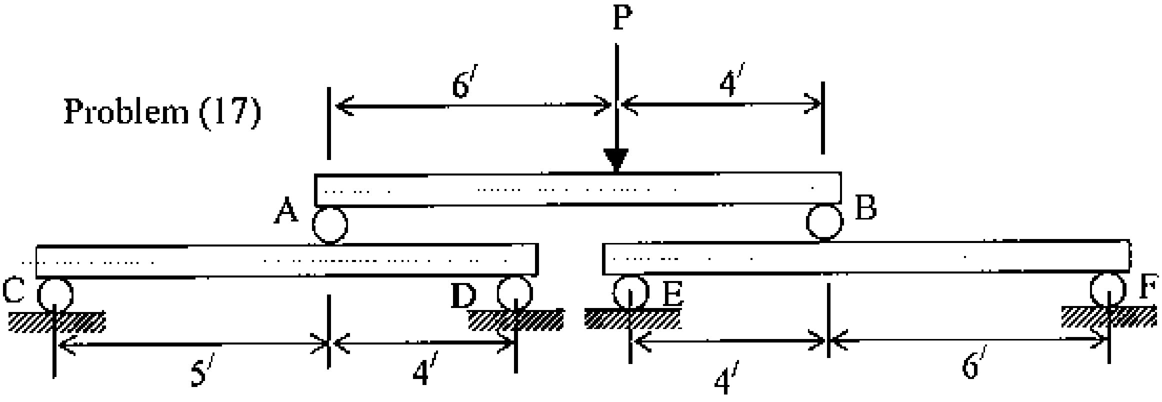
Problem (16): Draw free body diagrams of the beam AB and pulley C. Given, mass of the pulley 20 N and mass of the beam 2 N/m. Determine reactions at external supports A & B and internal support C.

Answer (16): $R_{AX} = 0$ N; $R_{AY} = 249.60$ N (↓); $R_{BY} = 797.60$ N (↑); $R_{CX} = 500$ N; $R_{CY} = 520$ N.



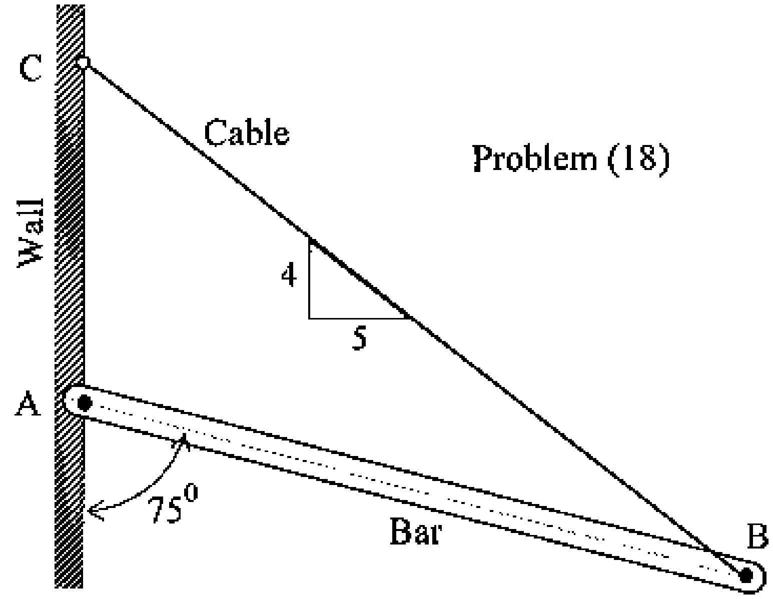
Problem (17): Determine the external support reactions at C, D, E & F and internal support reactions at A & B.

Answer (17): $R_{CY} = 8P/15$ (↑); $R_{DY} = 2P/9$ (↑); $R_{EY} = 9P/25$ (↑); $R_{FY} = 6P/25$ (↑); $R_{AY} = 4P/10$; $R_{BY} = 6P/10$.



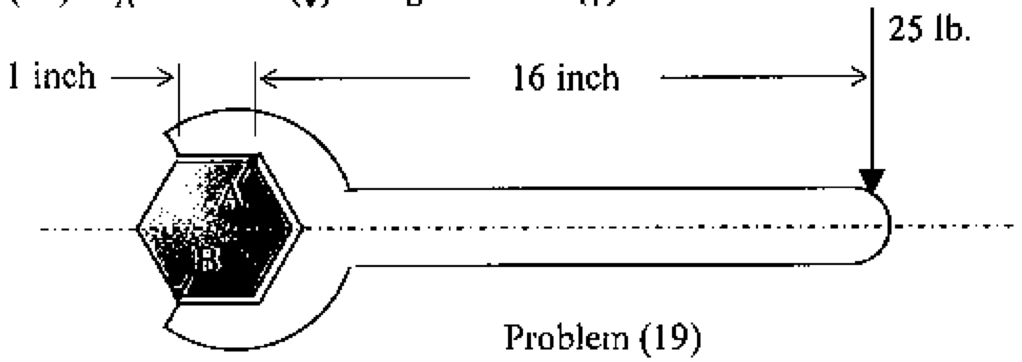
Problem (18): A bar of weight 100 lb is hinged to a vertical wall at A and supported by a cable. Determine cable force & components of pin reactions at A and C.

Answer (18): Force in cable = 120.3479846 lb. (tension); $R_{AX} = 93.97598742$ lb. (→); $R_{AY} = 24.81921006$ lb. (↑); $R_{CX} = 93.97598742$ lb. (←) & $R_{AY} = 75.18078994$ lb. (↑)



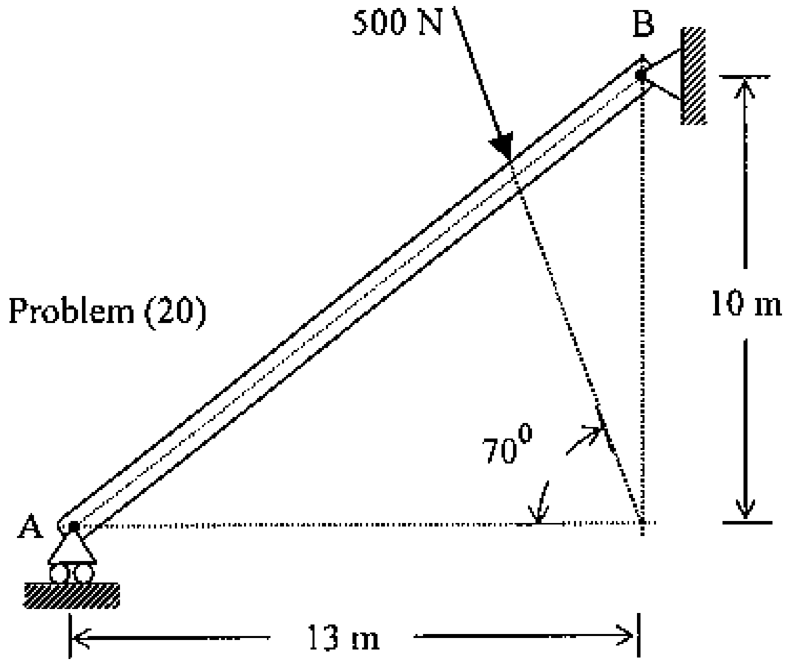
Problem (19): Determine the forces on the hex-head bolt at A and B when a vertical force of 25 lb is applied at end of the wrench, where the centerline is horizontal.

Answer (19): $R_A = 425$ lb. (↓) & $R_B = 400$ lb. (↑)



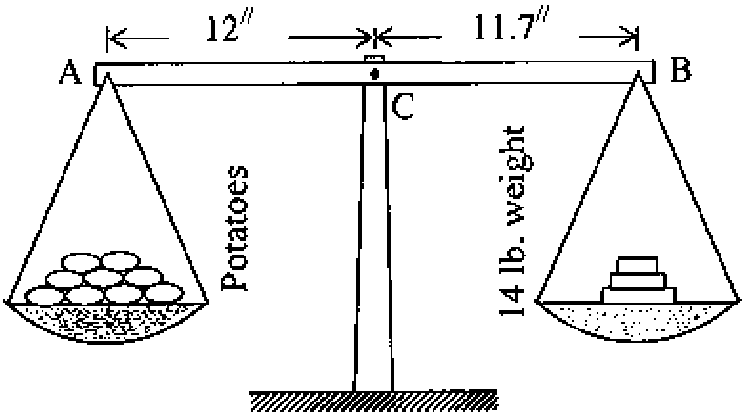
Problem (20): Determine support reactions at A & B.

Answer (20): R_{AY} = 131.546209 N (↑), R_{BX} = 171.0100717 N (←), R_{BY} = 338.3001014 N (↑)



Problem (21): A set of scales is faulty because the arms, instead of being the same length, are different. One is 11.7 inch and the other is 12 inch. A bag of potatoes hung on the longer arm is balanced by a weight of 14 lbs. on the other arm. What is its real weight?

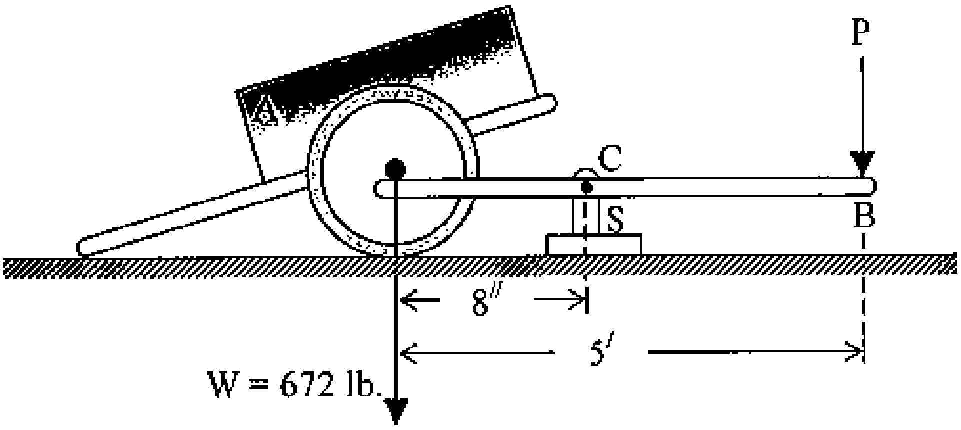
Answer (21): 13.65 lb.



Problem (21): Diagram illustrating faulty scales

Problem (22): To remove the wheels of a cart a lever "jack" is used. It consists of a strong wooden rod AB pivoted by a thin, strong metal pin C in a wooden stand S. The end A is placed below the cart axle and a force P is applied downwards at B to lift the cart bodily and enable a helper to place a support under the axle in order to keep it clear of the ground. If the jack rod is 5 ft. long, with the smaller arm AC 8 inch long, what force P must we exert at B to lift the cart, whose weight is 672 lb.?

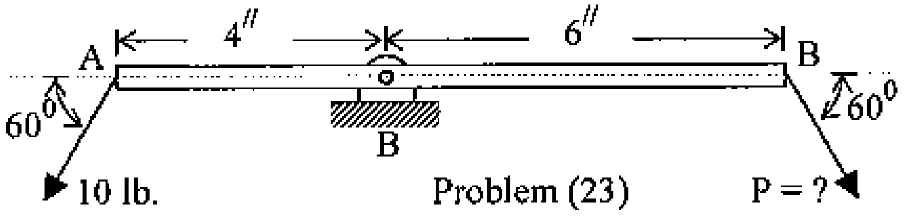
Answer (22): P = 103.3846154 lb. (↓)



Problem (22): Use of jack to remove wheels of a cart

Problem (23): What force P is needed to maintain equilibrium of bar ABC? Also determine the pin reaction at B.

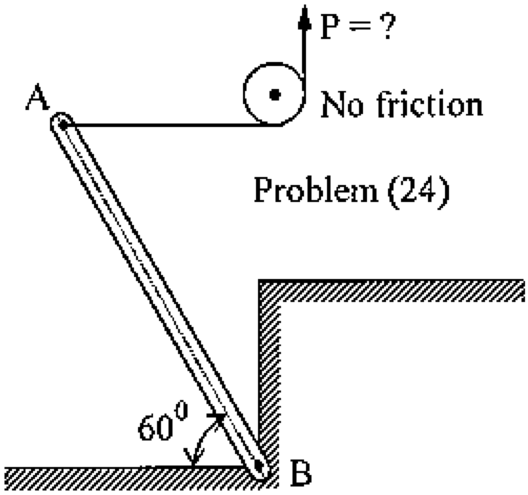
Answer (23): P = 6.666666667 lb.; R_{BX} = 1.666666667 lb. (→); R_{BY} = 14.43375673 lb. (↑)



Problem (23)

Problem (24): If there is no friction at the pulley, what force P is needed to keep the bar 60^0 align with horizontal (bar AB weighs 100 lb.)

Answer (24): P = 28.86751646 lb. (↑)



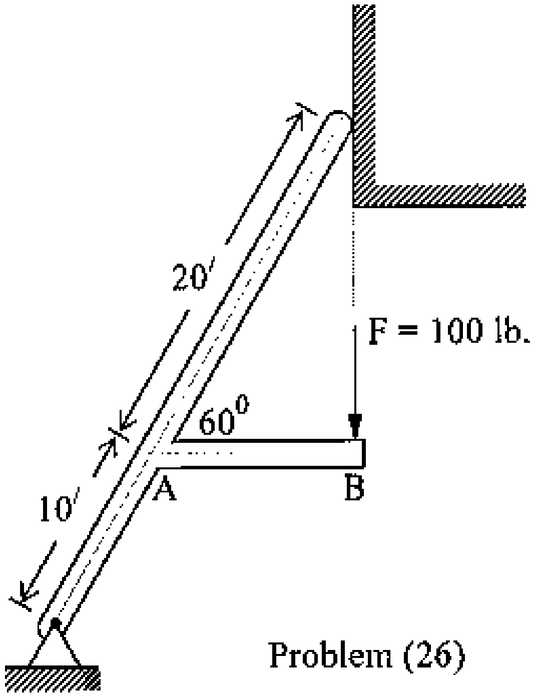
Problem (24)

Problem (25): A painter working on a wall is sitting on a plank 10 ft. long, suspended from ropes 6 inches from each end. He is seated 4.5 ft. from one end and his weight is 164 lbs. Neglecting the weight of the plank, what are the tensions in the two ropes?

Answer (25): $72\frac{8}{9}$ lbs. & $91\frac{1}{9}$ lbs.

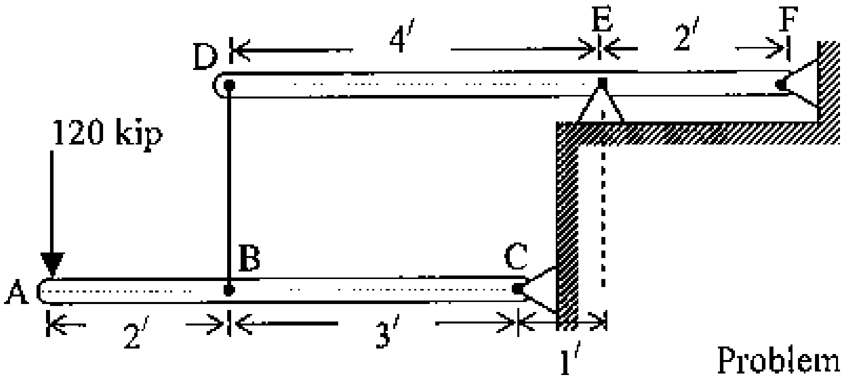
Problem (26): Determine the force in the lower support (a) for F acting at A & (b) for F acting at B.

Answer (26): (a) $\frac{100}{3\sqrt{3}}$ lb. (→) & 100 lb. (↑); (b) $\frac{100}{\sqrt{3}}$ lb. (→) & 100 lb. (↑)



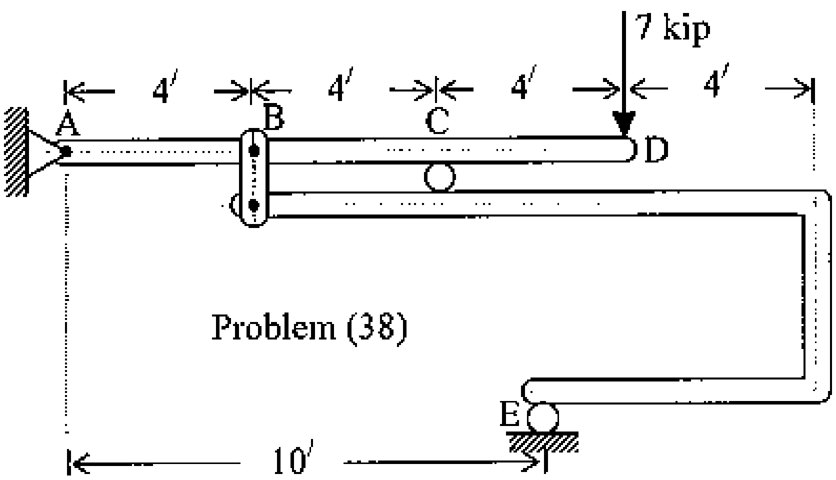
Problem (26)

Problem (37): Determine tension in the cable BD & vertical reaction at E.
Answer (37): T_{BD} = 200 kip & R_{EY} = 600 kip (↑)



Problem (37)

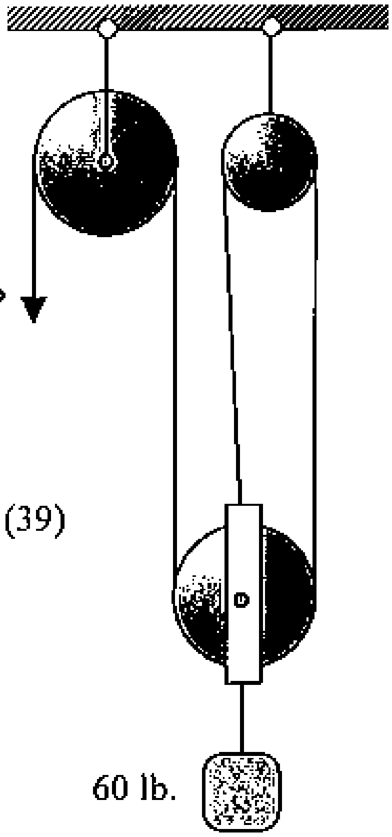
Problem (38): Determine internal support reaction at roller C.
Answer (38): 12.60 kip.



Problem (38)

Problem (39): What force P is needed to hold the 60 lb. weight in equilibrium? Assume ropes are parallel.

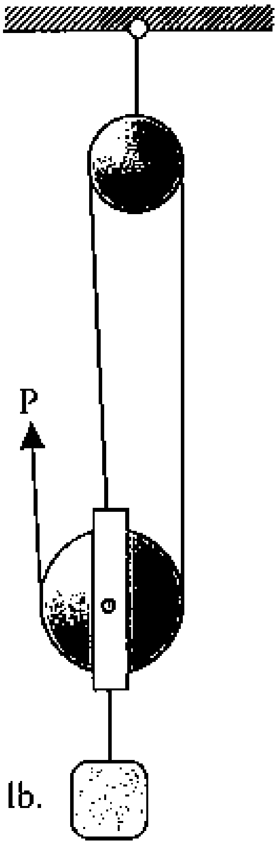
Answer (39): P = 20 lb.



Problem (39)

Problem (40): What force P is needed to hold the 60 lb. weight in equilibrium? Assume ropes are parallel.

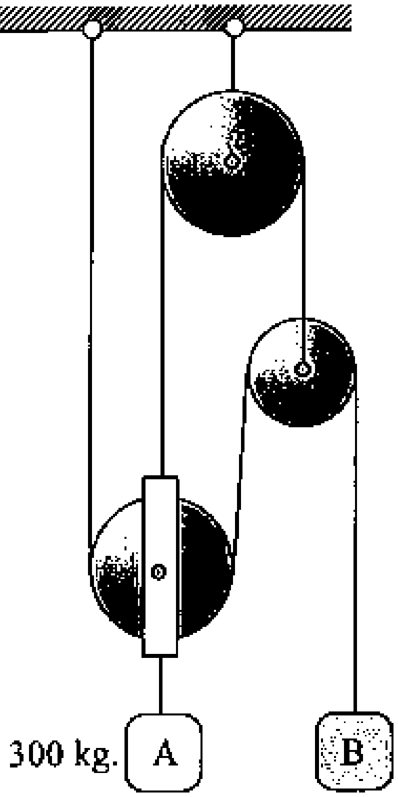
Answer (40): P = 20 lb.



Problem (40)

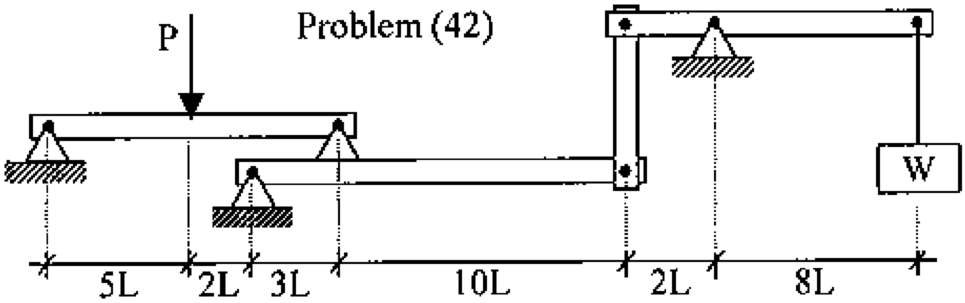
Problem (41): What weight B needed to maintain equilibrium
Answer (41): 75 kg.

Problem (41)

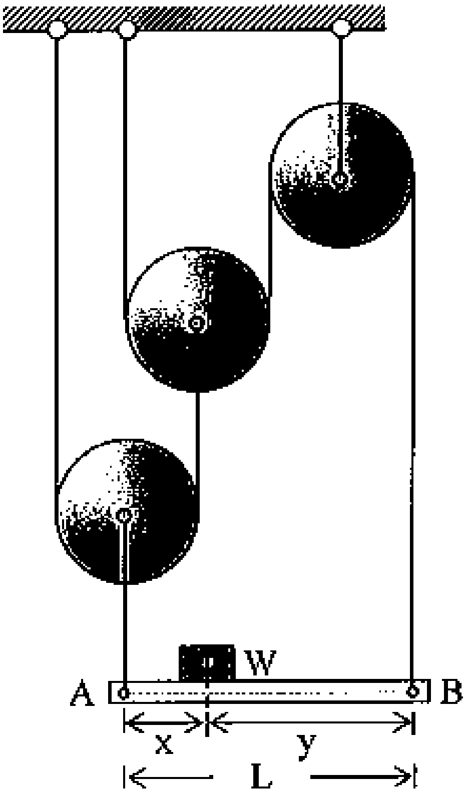


Problem (42): Express W as a function of P.
Answer (42): W = 3P/104

Problem (42)



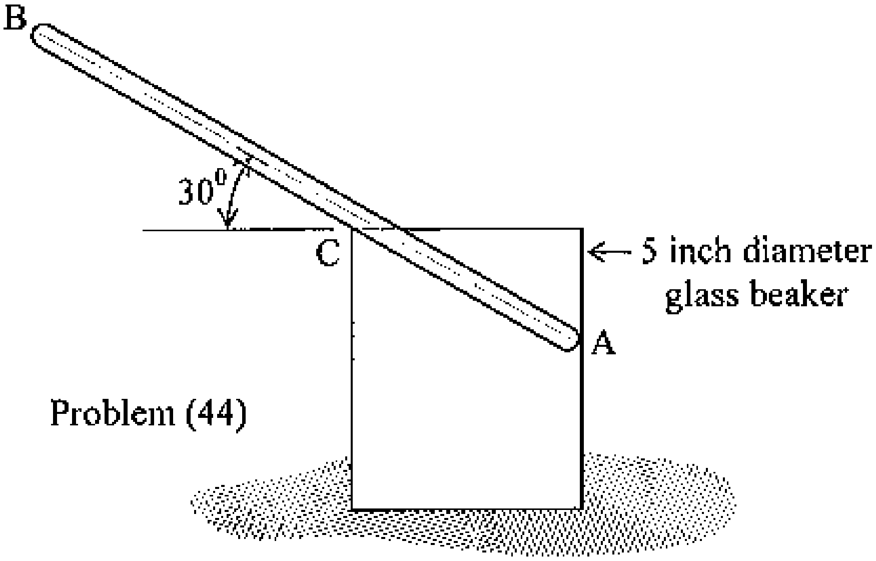
Problem (43): Locate the position of weight W to keep the bar AB in a horizontal position. Determine tension in the two ropes.
Answer (43): $x = L/5$ & $y = 4L/5$; $T_A = 4W/5$ & $T_B = W/5$.



Problem (43)

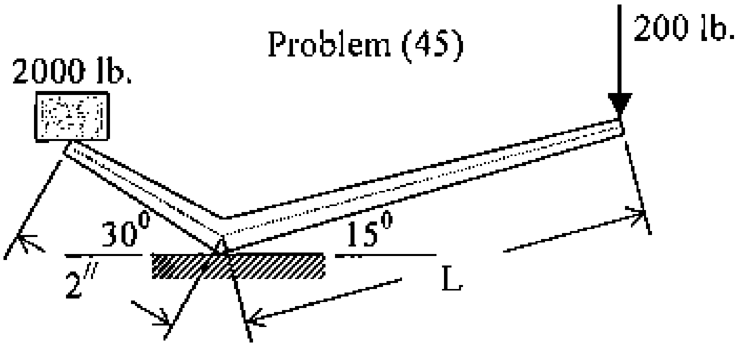
Problem (44): The glass rod AB of weight 8 gm. is placed in a glass beaker in a position of equilibrium. Considering all the surfaces to be smooth, determine the length of the glass rod and reactions at contact points A and C.

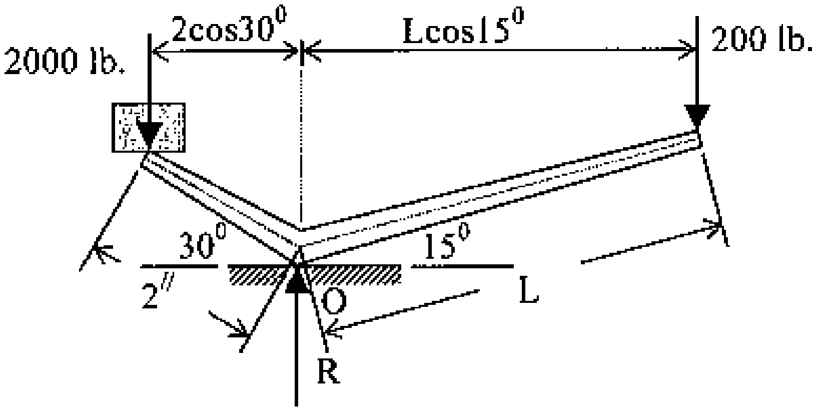
Answer (44): AB = 69.2820323 inch.; $R_A = \frac{8}{\sqrt{3}}$ gm. & $R_C = \frac{20}{\sqrt{3}}$ gm.



Problem (44)

Problem (45): A load of 2000 lb. needs to be raised by a crowbar. If a 200 lb. man rests his full weight, how long must dimension L be to accomplish this job? Assume all surfaces smooth.


Problem (45)



Solution (45): Free-boy of the crowbar

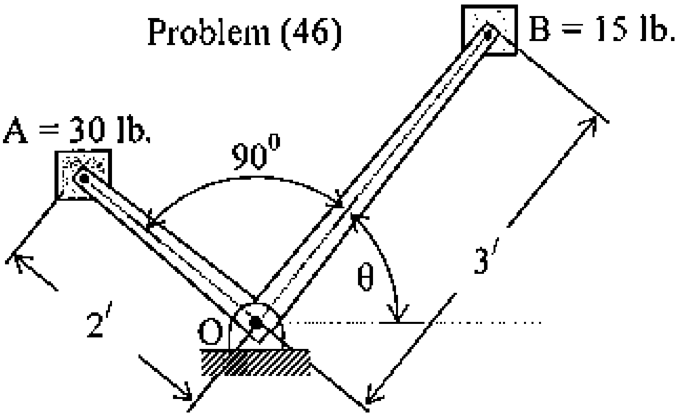
Solution (45): $\sum M_O = 0$ (clockwise +ve); $+ 200 \times L \cos 15^0 - 2000 \times 2 \cos 30^0 = 0$

$\Rightarrow L = 17.93150944$ inch.

Answer (45): 17.93150944 inch.

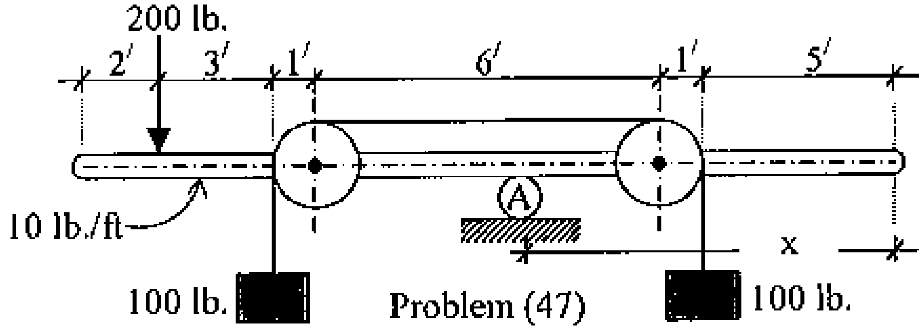
Problem (46): Two weights, A = 30 lb. and B = 15 lb., are attached to a bell crank whose weight is neglected. Determine the angle θ for equilibrium.

Answer (46): 36.86989765^0.


Problem (46)

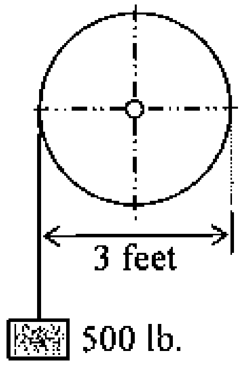
Problem (47): The arrangement may be balanced on roller A. Calculate the distance x for balance. Neglect the weights of the pulleys & rope.

Answer (47): x = 11.4137931 feet.



Problem (47)

Problem (48): A load of 500 lb. is suspended by a chord from a stationary disk. Replace the load on the disk by a force through the axis and a couple.
Answer (48): 500 lb. (↓) & 750 ft-lb. (anti-clockwise).



Problem (48)

Problem (49): Four soldiers, A, B, C & D stand 3 feet apart and on a 16 feet long uniform timber that weighs 500 lb. The soldiers weigh 150 lb., 175 lb., 200 lb., & 250 lb., respectively, A stands at one end of the timber. Find the location at which a fulcrum might be placed to balance the timber.
Answer (49): 6.254901961 feet from soldier A.

Problem (50): The resultant of two parallel forces which act in the same sense is 900 lb. The line perpendicular to the two force vectors is 9 feet between the force vectors and divided into parts in the ratio of 5 to 4 by the resultant vector. Compute the magnitude of the two parallel forces.
Answer (50): 500 lb. & 400 lb.

Problem (51): The resultant of two parallel forces, whose vectors are 10 feet apart and act opposite senses, is 700 lb. If the resultant acts 3 feet from one of the forces, what are the forces?
Answer (51): 910 lb. & 210 lb

Chapter Seven Centriods

Introduction: It has been established since long, that every particle of a body is attracted by the earth towards its centre. The force, with which a particle is attracted by the earth, is proportional to the mass of the particle and acts vertically downwards

$$W \propto m$$

$$W = mg$$

Where, W = weight of the body
m = mass of the body
g = acceleration due to gravity ($9.81\ m/sec^2$ or $32.2\ ft/sec^2$)

As the distance between the different particles of a body and the center of the earth is the same*, therefore these forces are taken to act along parallel lines. The point, through which the resultant of such parallel forces acts, is known as the **center of gravity** (briefly written as C.G.). It may be noted that every body has one, and only one, centre of gravity, irrespective of its position/orientation.

---

*Strictly speaking, this distance is not the same. But it is taken to be the same, because of very small size of the body as compared to that of the earth.

Centre of Gravity of a Plate: Consider a plate of uniform thickness 't' as shown in figure 1. Let it be divided into a number of elements. Let W be the total weight and W_i the weight of i th element of the plate located at the point (x_i, y_i) in the XY plane. Let $(\bar{x}, \bar{y})$ be the coordinates of center of gravity of plate. Then, total weight, $W = \sum W_i$.

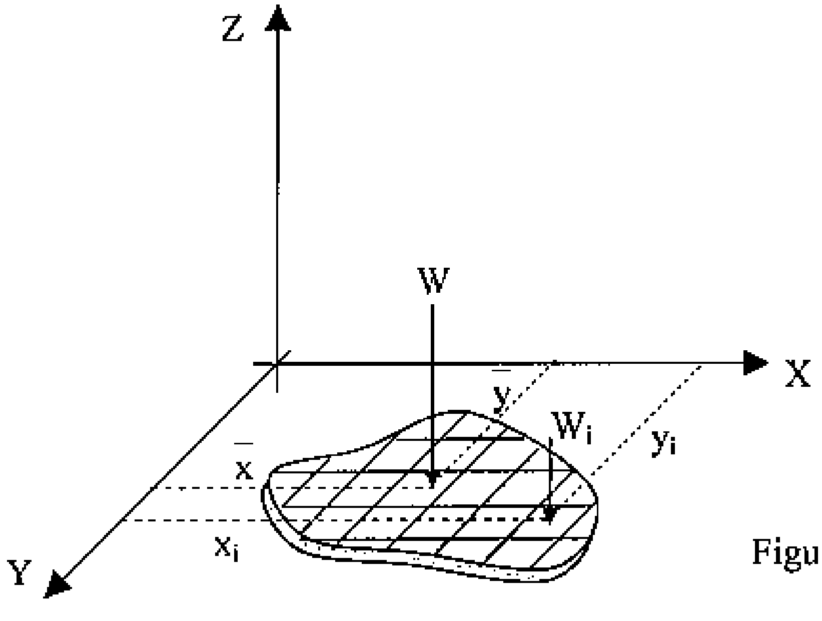


Figure 1

Equating the moment of total weight W about X-axis to the moment of its component weights, we get

$$W\bar{y} = W_1 y_1 + W_2 y_2 + W_3 y_3 + \dots\dots$$
$$= \sum W_i y_i$$
$$\bar{y} = \frac{\sum W_i y_i}{\sum W_i} = \frac{\sum W_i y_i}{W}$$

Similarly, considering moment about Y-axis, we get

$$W\bar{x} = W_1 x_1 + W_2 x_2 + W_3 x_3 + \dots\dots$$
$$= \sum W_i x_i$$
$$\bar{x} = \frac{\sum W_i x_i}{\sum W_i} = \frac{\sum W_i x_i}{W}$$

Experimental Determination of Centre of Gravity: See page no 111 of Introduction to Structural Mechanics (for building & architectural students) – Trefor J. Reynolds & Lewis E. Kent, The English Universities Press Ltd., St. Paul's House Warwick Lane, London EC4.

Centroid: Let γ be the unit weight of the material of plate, and t its uniform thickness. Let A_i be the area of i th plate.

$$W_i = \gamma A_i t$$
$$W = \sum W_i = \gamma t \sum A_i$$
$$= \gamma t A$$

Where, $A = \sum A_i$ is the total area of the plate

$$\bar{y} = \frac{\sum W_i y_i}{W} = \frac{\gamma t \sum A_i y_i}{\gamma t A} = \frac{\sum A_i y_i}{A}$$
$$\bar{x} = \frac{\sum W_i x_i}{W} = \frac{\gamma t \sum A_i x_i}{\gamma t A} = \frac{\sum A_i x_i}{A}$$

If the thickness of plate tends to zero, it reduces to an area. Hence, we can say the above two equations gives the location of center of gravity of a sectional area of a member. Thus, the center of gravity of a sectional area may be located from any axis by dividing moment of area about the axis by the total area. For the areas, the term centre of gravity is a misnomer. It should be called as centroid of the area.

It may be noted that since distance of centriod of an area from any axis is given by moment of area about the axis by the total area, the moment of area about the centroid axis is zero.

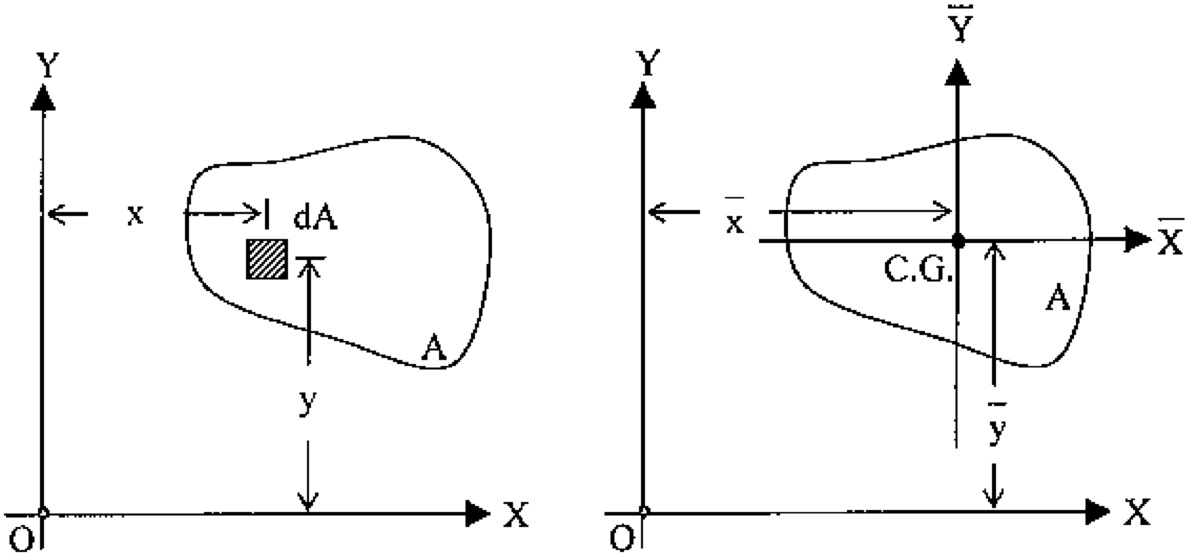


Figure 2: Definition of centroid of an area

Moment of area about X axis, $M_x = \int y\,dA = \bar{y}\,A$

Moment of area about Y axis, $M_y = \int x\,dA = \bar{x}\,A$

Co-ordinate of centroid = $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\int x\,dA}{A} \text{ and } \bar{y} = \frac{\int y\,dA}{A}$$

The concept of centroid may be extended to curved surfaces, lines and volumes in the following manner.

(a) Curved surface

$$\bar{x} = \frac{\int x\,dA}{\int dA};\ \bar{y} = \frac{\int y\,dA}{\int dA}$$

(b) Line

$$\bar{x} = \frac{\int x\,dL}{\int dL};\ \bar{y} = \frac{\int y\,dL}{\int dL}$$

(c) Volume

$$\bar{x} = \frac{\int x\,dV}{\int dV};\ \bar{y} = \frac{\int y\,dV}{\int dV}\ \&\ \bar{z} = \frac{\int z\,dV}{\int dV}$$

Difference between Centre of Gravity and Centroid: The differences between center of gravity and centroid may be noted.

(1) The term cente of gravity refers to the bodies with mass and weight, while centroid refers to the plane areas.

(2) Centre of gravity of a body is a point through which resultant of total weight acts for any orientation of the body and centroid refers to the point in the plane area that moment of area about any axis through that point is zero.

X = X-X axis
Y = Y-Y axis
$\overline{X} = \overline{X} - \overline{X}$ axis
$\overline{Y} = \overline{Y} - \overline{Y}$ axis
X-X axis & Y-Y axis are references axes
$\overline{X} - \overline{X}$ axis & $\overline{Y} - \overline{Y}$ axis are centroidal axes
x = distance from Y axis (or distance along X axis)
y = distance from X axis (or distance along Y axis)
$\overline{x}$ = distance of centriod from Y axis (abscissa of centroid from reference axis)
$\overline{y}$ = distance of centriod from X axis (ordinate of centroid from reference axis)

Axis of references: The center of gravity (C.G.) of a body is always calculated with reference to some assumed axis of reference (or sometimes with reference to some point of reference), there is no set rule for the selection of axis of reference. But the general practice is to take the axis of reference as the lowest line of the figure for calculating $\overline{y}$ and the left line of the figure for calculating $\overline{x}$. The location of the centroid in an area is independent of the reference axis employed.

Axis of symmetry: Sometimes the given section, whose C.G. is required to be found out, is symmetrical about X-X axis or Y-Y axis. In such cases, the procedure for calculating the C.G. is very much simplified, as we have only to calculate either $\overline{x}$ or $\overline{y}$. This is due to the reason that C.G. of the body will lie on the axis of symmetry.

If a plane figure has a line (or lines) of symmetry*, its centroid lies on that lines (or lines).
A plane of symmetry of a solid body contains the centroid.

*[If a plane area has an axis of symmetry, the centoid will lie on that axis of symmetry. This is so because the first moment about an axis of symmetry is zero (see figure 3(a)).
If a plane area has two axes of symmetry, the centroid of such an area will be at the intersection of the two axes of symmetry. Thus, the centroid of such an area can be located by inspection. Here, first moment of area about both axes will be zero (see figure 3(b)).]

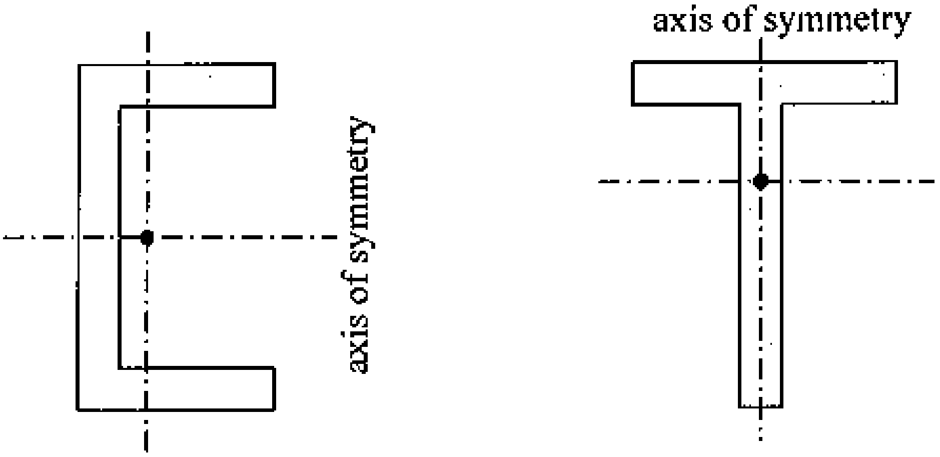


Figure 3(a): Area with one axis of symmetry

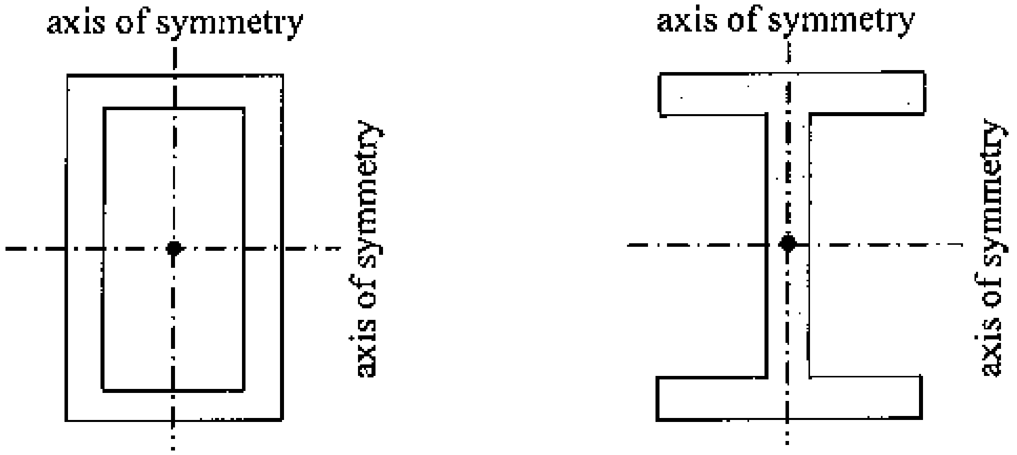


Figure 3(b): Area with two axes of symmetry

Centroid of Area Symmetrical about a Point: Figure 4 shows the special case of an area which is symmetric about a point. Such a point is called centre of symmetry and is so located that every line in the area drawn through that point is symmetric about that point. The centroid of such an area will lie at that point, which can be located by inspection.

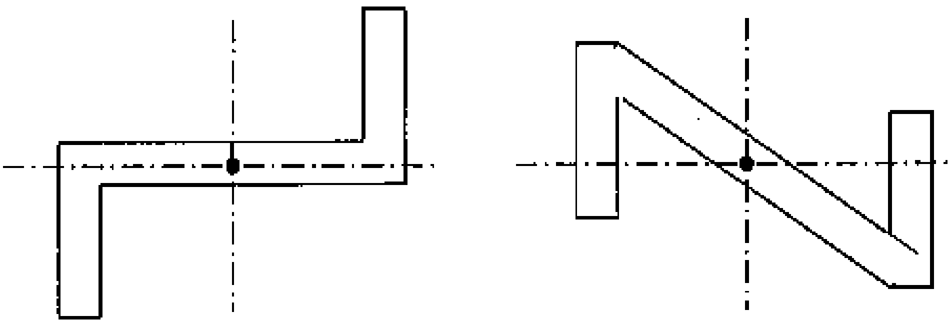


Figure 4: Area symmetric about a point.

Uses of Axis of Symmetry in Locating Centriod:

Axis of symmetry	Co-ordinate of centroid ($\bar{x},\bar{y}$)
If $\overline{X}$ (= $\overline{X}-\overline{X}$ axis)is axis of symmetry	$\bar{y}=0,\ \bar{x}=?$
If $\overline{Y}$ (= $\overline{Y}-\overline{Y}$ axis)is axis of symmetry	$\bar{x}=0,\ \bar{y}=?$
If $\overline{X}$ (= $\overline{X}-\overline{X}$ axis)and $\overline{Y}$ (= $\overline{Y}-\overline{Y}$ axis)are both axes of symmetry	$\bar{x}=0,\ \bar{y}=0$

If a plane area has an axis of symmetry, the centroid will lie on that axis of symmetry. This is so because the first moment about an axis of symmetry is zero. In such a case, only one distance need be calculated.

If a plane area has two axes of symmetry, the centroid of such an area will be lie at the intersection of the axes of symmetry. Thus, the centroid of such an area can be located by inspection. Hence first moment of area about both the axes will be zero.

Determination of Centroid of Simple Figures

- Centriod of a Rectangle
- Centriod of a Triangle: $\frac{2h}{3}$ from apex or $\frac{h}{3}$ from base, where h is the height of the triangle.
- Centriod of an Arc of a Circle: $\frac{r\sin\beta}{\beta}$ from centre of arc along its axis of symmetry, where, r is the radius of arch; 2β is the total angle subtended at centre i.e. β is the half of the total angle subtended at center; β is in radian.
- Centriod of a Sector of a Circle: $\frac{2r\sin\beta}{3\beta}$ from centre of circle along its axis of symmetry, where, r is the radius of circle; 2β is the total angle subtended at centre i.e. β is the half of the total angle subtended at center; β is in radian.
- Centriod of a Parabolic Spandrel:
- Centriod of a Right Circular Cone: $\frac{3h}{4}$ from apex or $\frac{h}{4}$ from base, where h is the height of the cone.
- Centriod of a Hemi-Sphere: $\frac{3r}{8}$ from base or $\frac{5r}{8}$ from crest, where r is the radius of sphere.

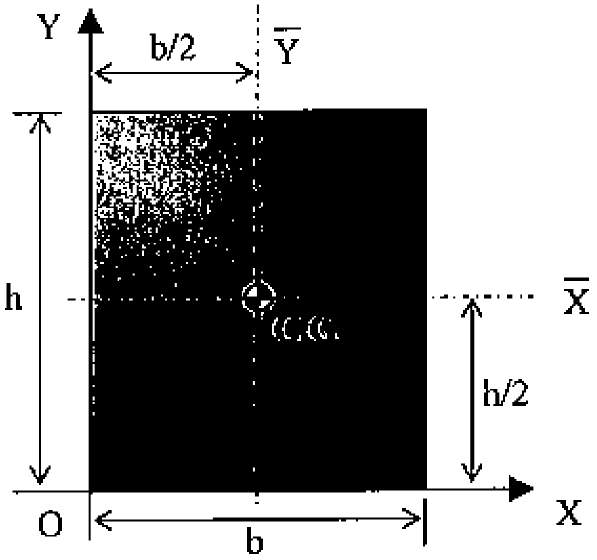


Figure 5(a): Centroid of a rectangle

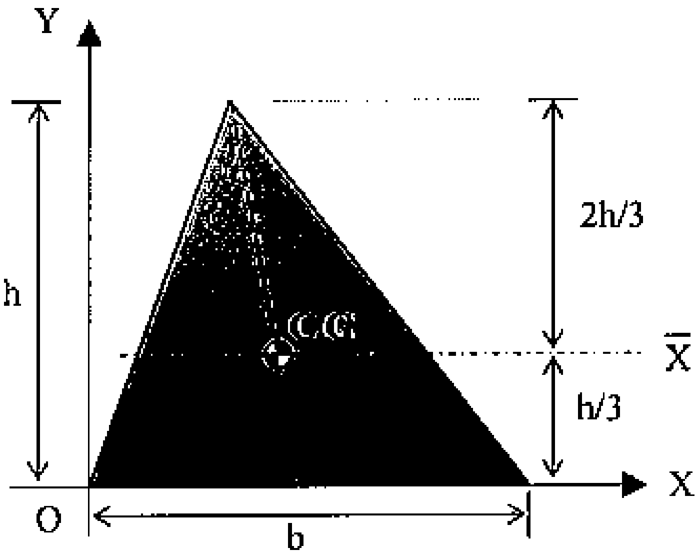


Figure 5(b): Centroid of a triangle

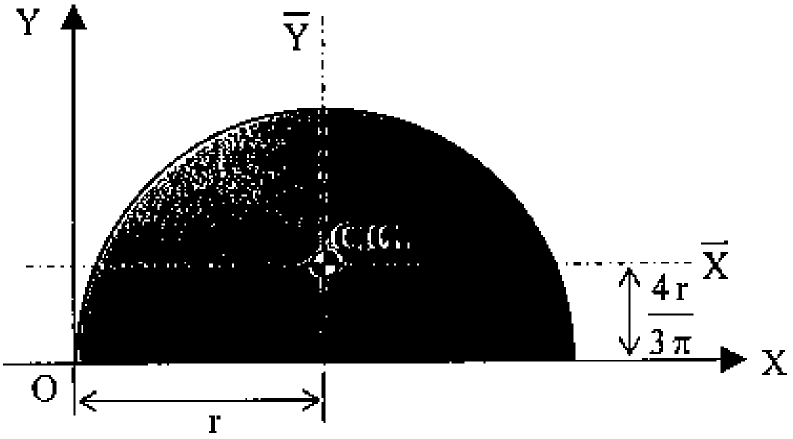


Figure 5(c): Centroid of a semi-circular area

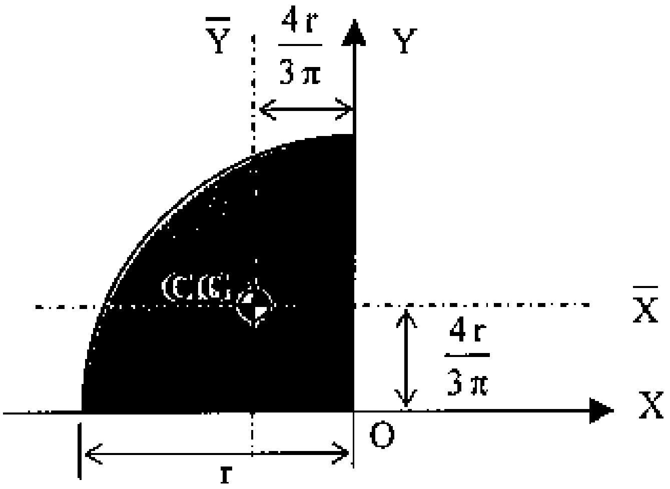


Figure 5(d): Centroid of a quarter-circular area

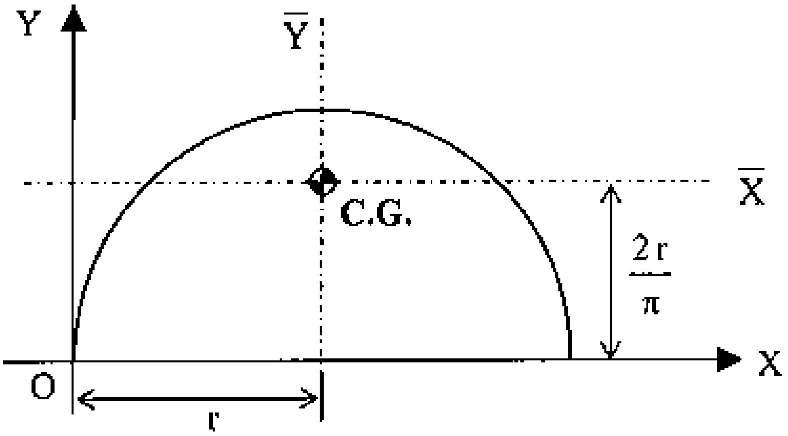


Figure 5(e): Centroid of a semi-circular arc

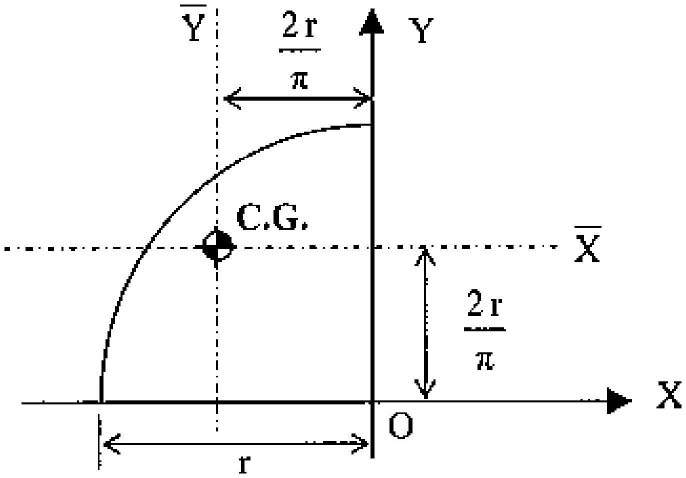


Figure 5(f): Centroid of a quarter-circular arch

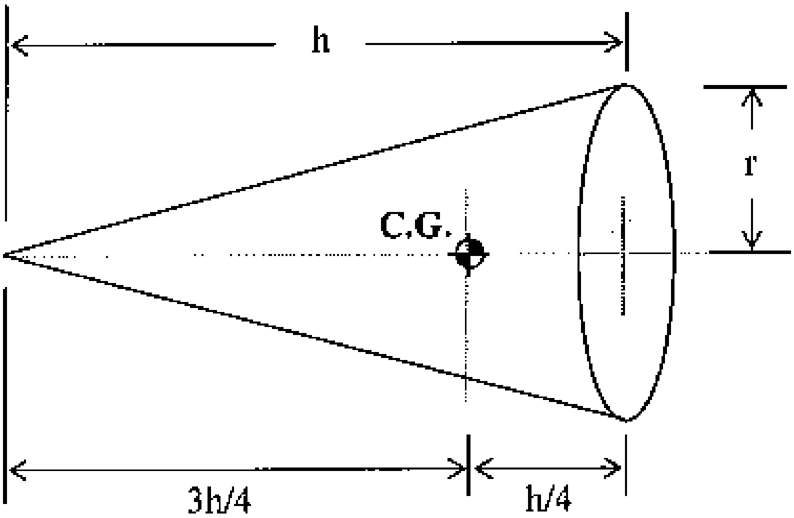


Figure 5(g): Centroid of a right circular cone

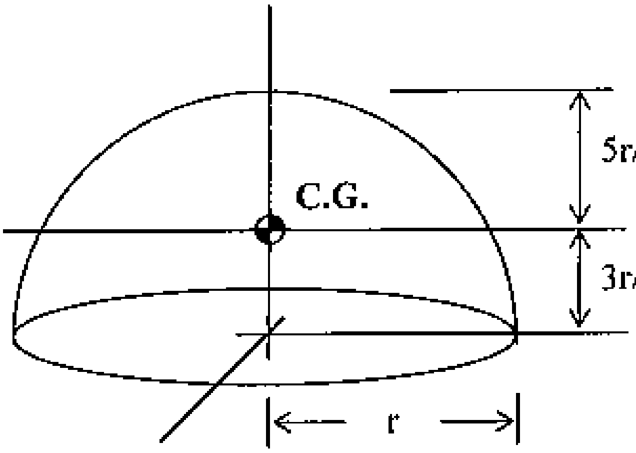


Figure 5(h): Centriod of a hemi-sphere

Two methods are available in computing centroid:
(1) Method of Integration
(2) Composite section / Method of summation

Centroid required for	Differential element	
	Integration method	Summation method
1-D figure (line)	Differential line (dL)	Differential line (ΔL)
2-D figure (area)	Differential area (dA)	Differential area (ΔA)
3-D figure (volume)	Differential volume (dV)	Differential volume (ΔV)

Centroid by Integration:

(a) Curved surface

$$\bar{x} = \frac{\int x\, dA}{\int dA}; \ \bar{y} = \frac{\int y\, dA}{\int dA}$$

(b) Line

$$\bar{x} = \frac{\int x\,dL}{\int dL};\ \bar{y} = \frac{\int y\,dL}{\int dL}$$

(c) Volume

$$\bar{x} = \frac{\int x\,dV}{\int dV};\ \bar{y} = \frac{\int y\,dV}{\int dV}\ \&\ \bar{z} = \frac{\int z\,dV}{\int dV}$$

(d) Mass

$$\bar{x} = \frac{\int x\,dm}{\int dm};\ \bar{y} = \frac{\int y\,dm}{\int dm}\ \&\ \bar{z} = \frac{\int z\,dm}{\int dm}$$

(e) Weight

$$\bar{x} = \frac{\int x\,dW}{\int dW};\ \bar{y} = \frac{\int y\,dW}{\int dW}\ \&\ \bar{z} = \frac{\int z\,dW}{\int dW}$$

Centroid of Composite Section: Method of locating centroid from first principle (method of integration) is convenient only in case of simple figures where a general element can be taken and integration is carried out to cover the entire area. But in practice, we need centroid of composite sections which are built up with many silmple figures. For locating centroid of such sections, following steps may be followed:

(1) Divide the composite section into a number of simple figures like rectangles, triangles, circle (quarter circle, semi-circle, full circle).
(2) Locate the centroid of each simple figure.
(3) Find the total area of composite section by adding areas of simple figures.
(4) Assume area of simple figure is concentrated at its centroid, find the moment of area of each simple figure about the axis considered. Sum up moment of areas of simple figures to get moment of area of the composite figure.
(5) Then the distance of the centroid of the composite figure is obtained by diving moment of area of the composite section by total area of the composite section.

▶ Composite area

$$\bar{x} = \frac{\sum Ax}{\sum A} = \frac{A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + \dots\dots + A_nx_n}{A_1 + A_2 + A_3 + A_4 + \dots\dots + A_n}$$

$$\bar{y} = \frac{\sum Ay}{\sum A} = \frac{A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4 + \dots\dots + A_ny_n}{A_1 + A_2 + A_3 + A_4 + \dots\dots + A_n}$$

▶ Composite line

$$\bar{x} = \frac{\sum Lx}{\sum L} = \frac{L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4 + \ldots\ldots\ldots\ldots + L_nx_n}{L_1 + L_2 + L_3 + L_4 + \ldots\ldots\ldots + L_n}$$

$$\bar{y} = \frac{\sum Ly}{\sum L} = \frac{L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4 + \ldots\ldots\ldots\ldots + L_ny_n}{L_1 + L_2 + L_3 + L_4 + \ldots\ldots\ldots + L_n}$$

▶ Composite volume

$$\bar{x} = \frac{\sum Vx}{\sum V} = \frac{V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + \ldots\ldots\ldots\ldots + V_nx_n}{V_1 + V_2 + V_3 + V_4 + \ldots\ldots\ldots + V_n}$$

$$\bar{y} = \frac{\sum Vy}{\sum V} = \frac{V_1y_1 + V_2y_2 + V_3y_3 + V_4y_4 + \ldots\ldots\ldots\ldots + V_ny_n}{V_1 + V_2 + V_3 + V_4 + \ldots\ldots\ldots + V_n}$$

▶ Composite mass

$$\bar{x} = \frac{\sum mx}{\sum m} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + \ldots\ldots\ldots\ldots + m_nx_n}{m_1 + m_2 + m_3 + m_4 + \ldots\ldots\ldots + m_n}$$

$$\bar{y} = \frac{\sum my}{\sum m} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 + \ldots\ldots\ldots\ldots + m_ny_n}{m_1 + m_2 + m_3 + m_4 + \ldots\ldots\ldots + m_n}$$

▶ Composite mass

$$\bar{x} = \frac{\sum Wx}{\sum W} = \frac{W_1x_1 + W_2x_2 + W_3x_3 + W_4x_4 + \ldots\ldots\ldots\ldots + W_nx_n}{W_1 + W_2 + W_3 + W_4 + \ldots\ldots\ldots + W_n}$$

$$\bar{y} = \frac{\sum Wy}{\sum W} = \frac{W_1y_1 + W_2y_2 + W_3y_3 + W_4y_4 + \ldots\ldots\ldots\ldots + W_ny_n}{W_1 + W_2 + W_3 + W_4 + \ldots\ldots\ldots + W_n}$$

Theorem of Pappus and Guldinus: The theorems of Pappus – Guldinus were first set forth by Pappus about 300 A. D. and then later restated by the Swiss mathematician Paul Habakuk Guldin about 1640. These theorems are concerned with surfaces of revolution as they relate to the generating curve and volumes of revolution as they relate to the generating area.

[Papus – lived at the end of the third century in Alexandria]

[Guldin – born in 1577 at Saint Gall, Switzerland & died in 1643 at Graz, Austria]

Theorem I: If a surface of revolution is generated by a plane curve revolving about any non-intersecting axis in its plane, the area of the generated surface is equal to the length of the curve times the distance traveled by the centroid of the curve while the surface is being generated.

$$dA = \theta\, y\, dL$$

$$\int dA = \int \theta\, y\, dL$$

$$A = \int dA = \int \theta\, y\, dL = \theta \int y\, dL$$

$$A_x = \theta\, \bar{y}\, L \text{ (for revolution about X axis)}$$

$$A_y = \theta\, \bar{x}\, L \text{ (for revolution about Y axis)}$$

Theorem II: If a volume of revolution is generated by a plane area revolving about any non-intersecting axis in the plane of the area, the generated volume is equal to the magnitude of the area times the distance traveled by the centroid of the area while the volume is being generated.

$$dV = \theta\, y\, dA$$

$$\int dV = \int \theta\, y\, dA$$

$$V = \int dV = \int \theta\, y\, dA = \theta \int y\, dA$$

$$V_x = \theta\, \bar{y}\, A \text{ (for revolution about X axis)}$$

$$V_y = \theta\, \bar{x}\, A \text{ (for revolution about Y axis)}$$

Stability: A body is said to be stable when, after a slight displacement it returns to its original position. Thus, if the cone (figure 6(a)), be tilted slightly about the point O, and then left to itself, it at once returns to its original position. If, however, the cone is first carefully balanced about the vertex P, as in figure 6(b) and then slightly displaced, it will tumble down. This is an example of instability. In both cases the body was initially in equilibrium. We speak of the first as a case of stable equilibrium, and of the second equilibrium. The condition for stable equilibrium is that the C.G. shall be in the lowest possible position, so that the effect of a slight displacement is to the raise the C.G. as shown in the figure. For unstable equilibrium, however, the C.G. must be in the highest possible position, so that a slight displacement lowers the C.G.

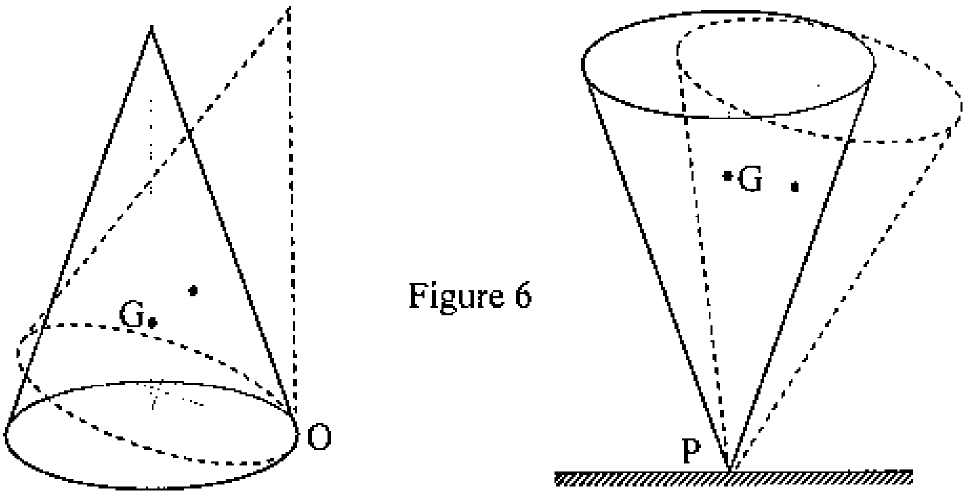
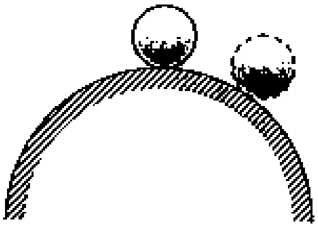


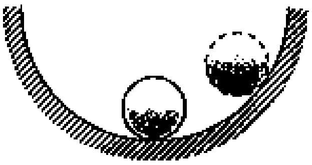
Figure 6

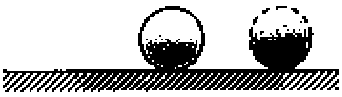
(a) Stable equilibrium of a Cone (b) Unstable equilibrium of a Cone

Jugglers and acrobats depend entirely upon their skill in balancing both themselves and various "props" in a condition of unstable equilibrium in order to gain the applause of their audiences.

In designing machinery the conditions for stability are very important. In a tramcar, for example, the weight is concentrated as near the ground as possible so that accidental "jumping" or "swaying" need not result in the overturning of the car, as would, of course, otherwise happen.


(a) unstable equilibrium


(b) stable equilibrium

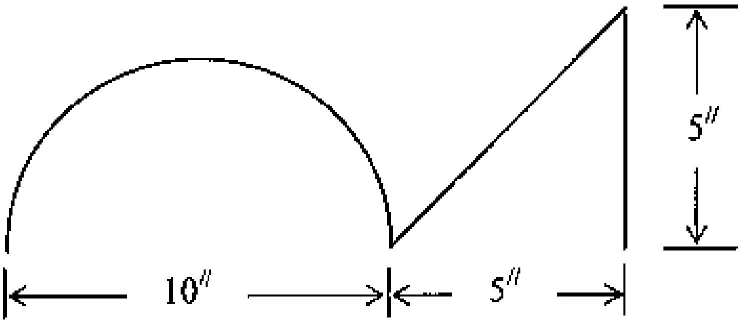

(c) neutral equilibrium

Finally, it is possible to have a body in a position such that a slight displacement will cause it neither to topple nor to return to its original position; it merely stays in its new position. A Cylinder on its curved surface or a ball resting on the ground is an example of this. We speak of this as neutral equilibrium. The condition is that the C.G. shall remain at a constant distance above the ground when the body is displaced.

Problems

Problem (1): Determine the location of centroid of line.

Answer (1): $\bar{x} = 8.709020963''$ (from left); $\bar{y} = 2.886265878''$ (from bottom)

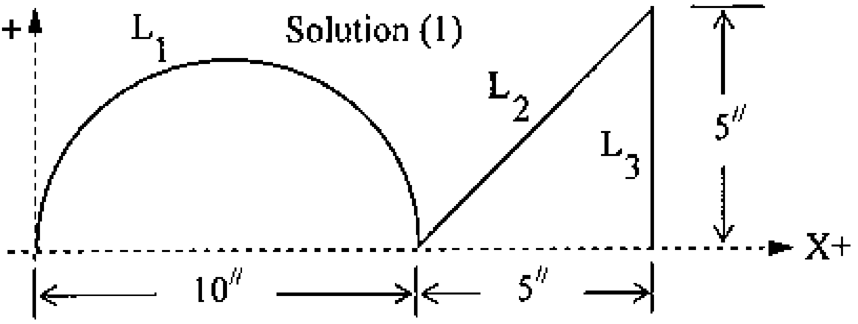


Problem (1)

Solution (1): $L_1 = \frac{2\pi r}{2} = \frac{2\pi(5)}{2} = 5\pi$ inch; $x_1 = 5$ inch; $y_1 = \frac{r\sin\beta}{\beta} = \frac{5\sin(\pi/2)}{(\pi/2)} = \frac{10}{\pi}$ inch.

$L_2 = \sqrt{(5)^2 + (5)^2} = 5\sqrt{2}$ inch; $x_1 = (10 + 5/2)$ inch = 12.5 inch; $y_1 = 5/2$ inch = 2.5 inch.

$L_3 = 5$ inch; $x_3 = 15$ inch; $y_3 = 5/2$ inch = 2.5 inch.



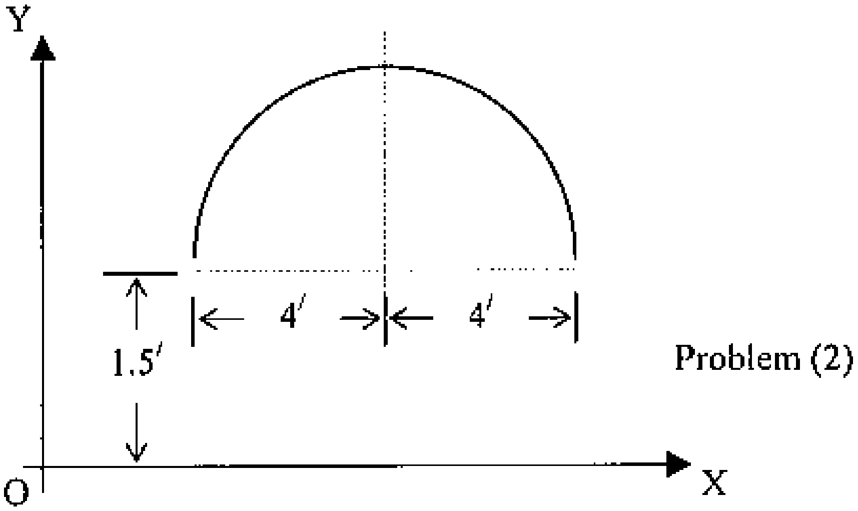
$$\bar{x} = \frac{\Sigma Lx}{\Sigma L} = \frac{L_1x_1 + L_2x_2 + L_3x_3}{L_1 + L_2 + L_3} = \frac{(5\pi)\times(5) + (5\sqrt{2})\times(12.5) + (5)\times(15)}{(5\pi + 5\sqrt{2} + 5)} =$$

8.709020963″ (from left).

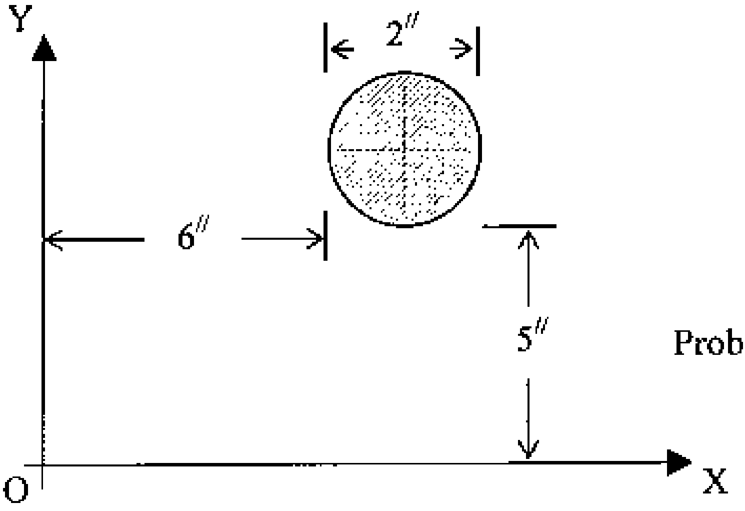
$$\bar{y} = \frac{\Sigma Ly}{\Sigma L} = \frac{L_1y_1 + L_2y_2 + L_3y_3}{L_1 + L_2 + L_3} = \frac{(5\pi)\times(10/\pi) + (5\sqrt{2})\times(2.5) + (5)\times(2.5)}{(5\pi + 5\sqrt{2} + 5)}$$

= 2.886265878″ (from bottom)

Problem (2): Determine the area generated by revolving the semi-circular arc through 270⁰ about X-axis.
Answer (2): 239.622880 square feet.



Problem (3): Determine the volume of torus generated by revolving the circular area through 360⁰ about X-axis and Y-axis.
Answer (3): 118.4352528 inch^3 & 138.1744616 inch^3.



Problem (3)

Solution (3): $A = \frac{\pi d^2}{4} = \frac{\pi(2)^2}{4} = \pi \text{ inch}^2$

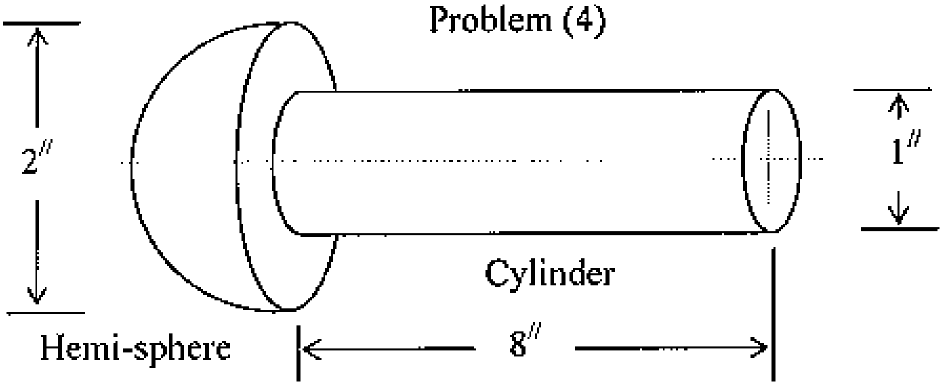
$\bar{x} = (6 + 2/2) = 7$ inch; $\bar{y} = (5 + 2/2) = 6$ inch; $\theta = 360^0 = 2\pi$ radian

$V_x = \theta\,\bar{y}\,A$ (for revolution about X axis) $= 2\pi \times 6 \times \pi = 118.4352528 \text{ inch}^3$

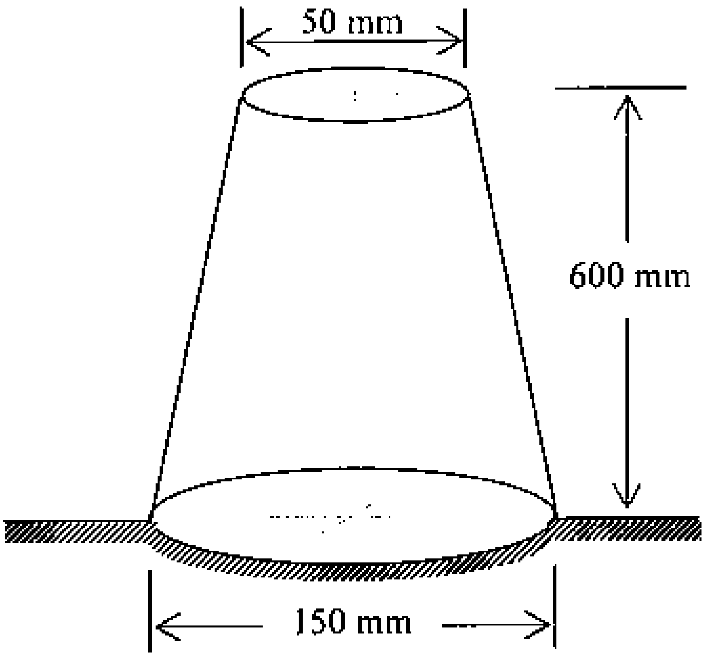
$V_y = \theta\,\bar{x}\,A$ (for revolution about Y axis) $= 2\pi \times 7 \times \pi = 138.1744616 \text{ inch}^3$

Problem (4): Determine the location of c.g. for the composite volume.
Answer (4): 4.336538462 inch (from right) or 4.663461538 inch (from left)

Problem (4)

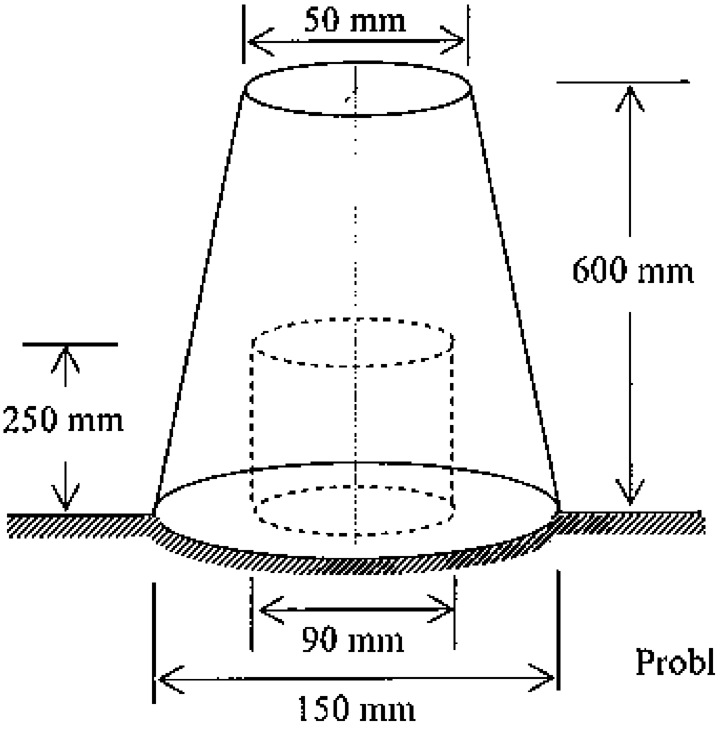


Problem (5): Determine the location of c.g. for the frustum.
Answer (5): 207.6923077 mm (from base)



Problem (5)

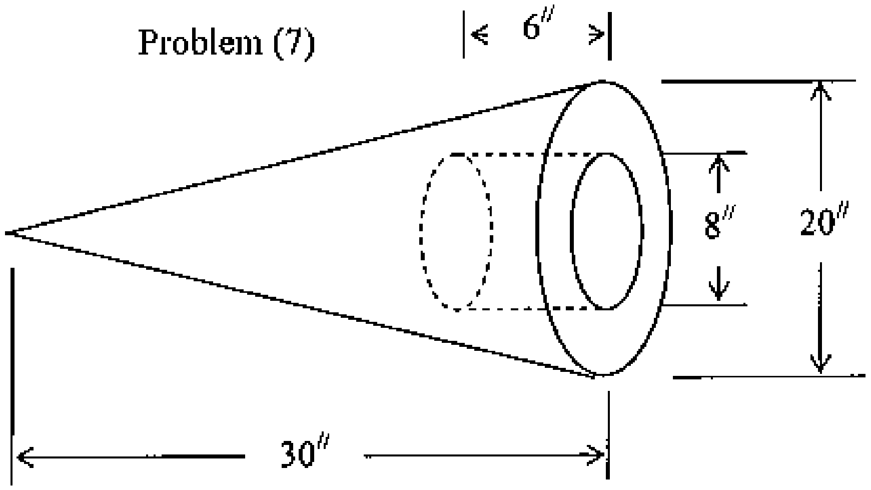
Problem (6): A cylindrical hole is made within a frustum. Locate center of gravity.
Answer (6): 310.3448276 mm (from base)



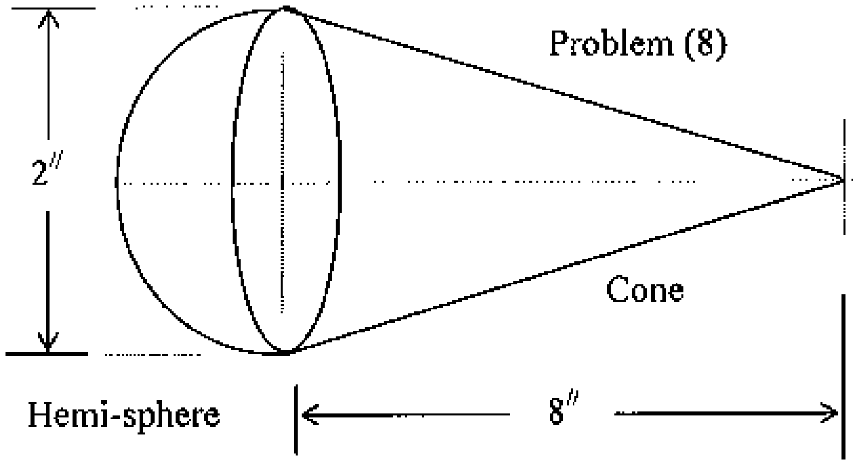
Problem (6)

Problem (7): A Cylindrical hole is made within a right circular cone. Locate the center of gravity of the cone with the hole.
Answer (7): 22.02212389 inch (from apex) or 7.97787611 inch (from base)

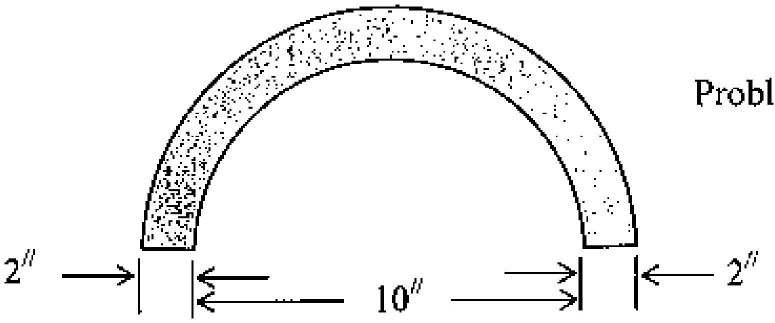
Problem (7)



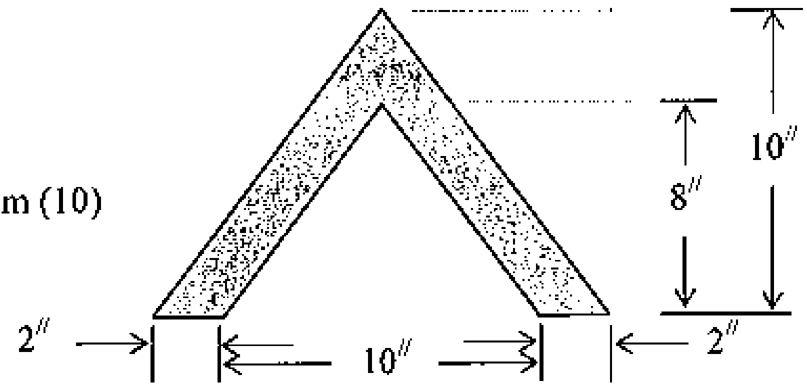
Problem (8): A hemisphere is attached to a right circular cone. Determine the location of centroid.
Answer (8): 2.525 ft (from left) or 6.475 ft (from right)



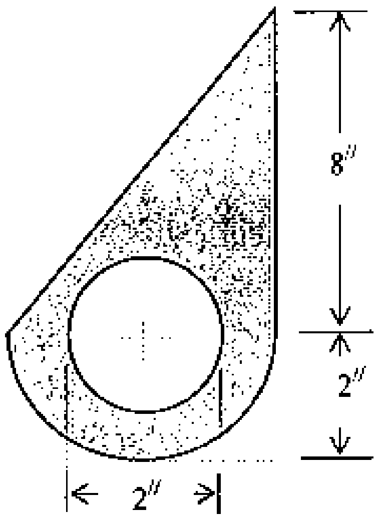
Problem (9) – (21): Determine Centroid of composite figures.
Answer (9): 3.855086399 inch (from base) & 7 inch (from right)
Answer (10): 7 inch (from left) & 4.2222222 inch (from base)
Answer (11): 1.4427496 inch (from left) & 6.04962241 inch (top)
Answer (12): 3.183483328 inch (from left) & 3.612372452 inch (from bottom)
Answer (13): 4.928945604 inch (from left) & 2.800939517 inch (from bottom)
Answer (14): 9.320783804 inch (from left) & 3.343451468 inch (from bottom)
Answer (15): 6 inch (from left) & 3.906604383 inch (from bottom)
Answer (16): 3.75 inch (from left) & 3.50 inch (from bottom)
Answer (17): 6.172850063 inch (from left) & 6 inch (from bottom)
Answer (18): 2.323852896 inch (from left) & 5 inch (from bottom)
Answer (19): 3.19971883 inch (from left) & 4 inch (from bottom)
Answer (20): 4.66500884 inch (from left)
Answer (21): 5.881091668 inch (from left) & 4.343869032 inch (from bottom)
Answer (22): 10 inch (from left) & 9.350387019 inch (from bottom)



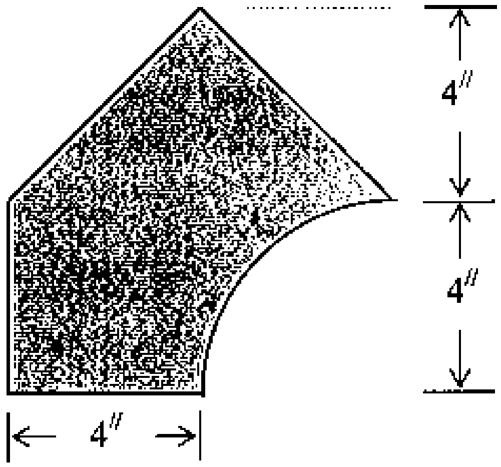
Problem (9)



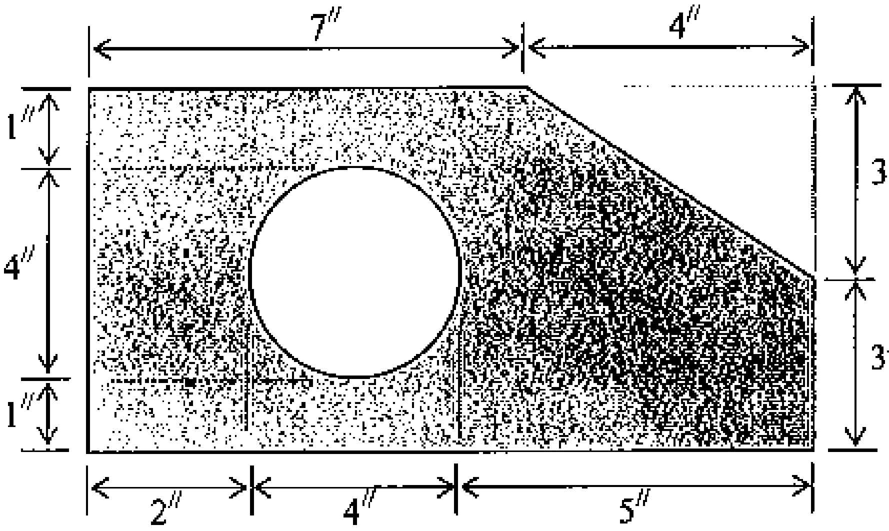
Problem (10)



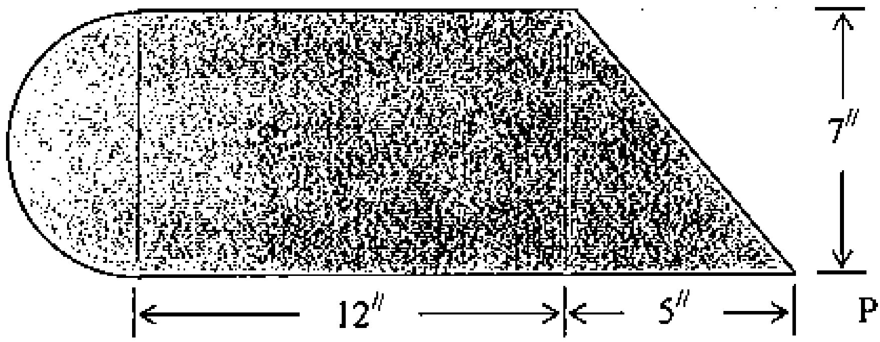
Problem (11)



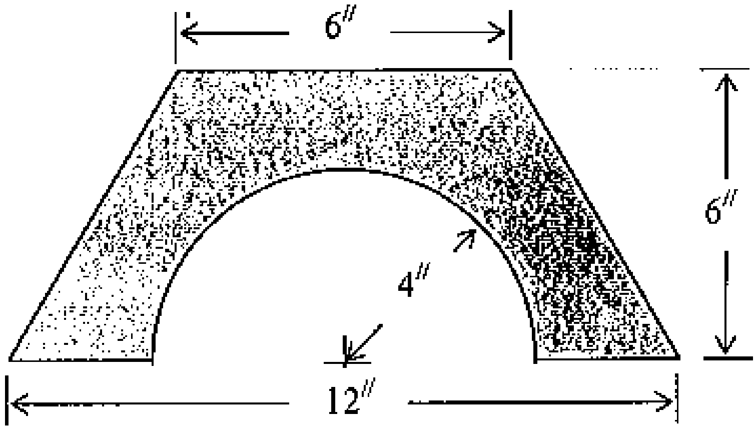
Problem (12)



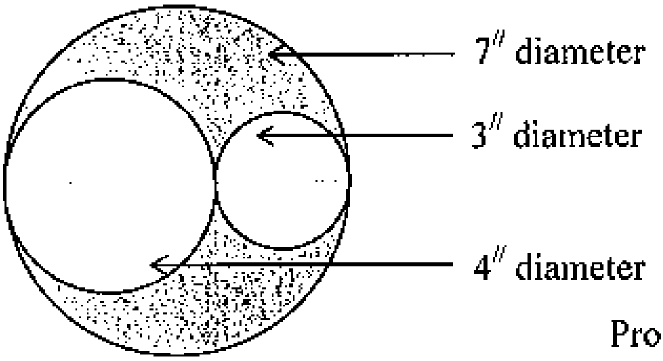
Problem (13)



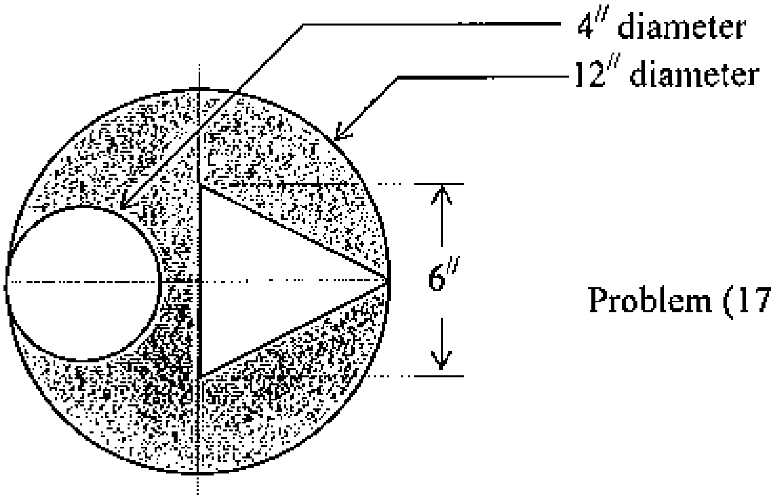
Problem (14)



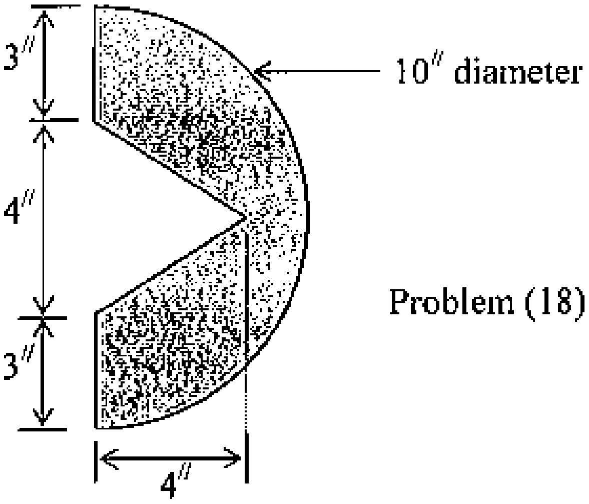
Problem (15)



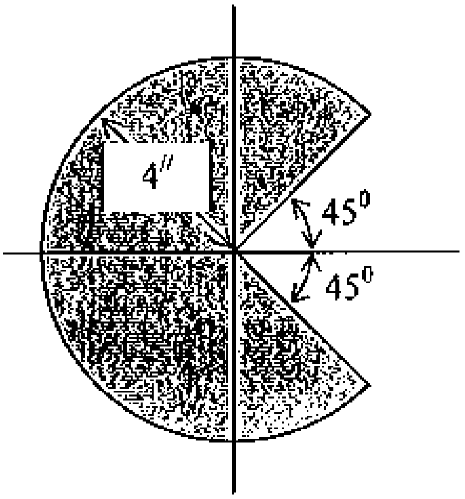
Problem (16)



Problem (17)

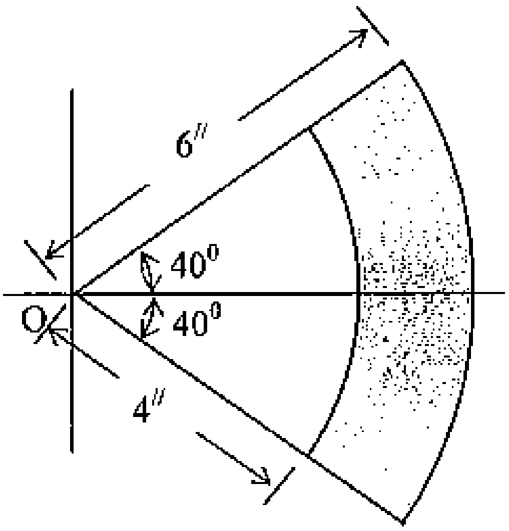


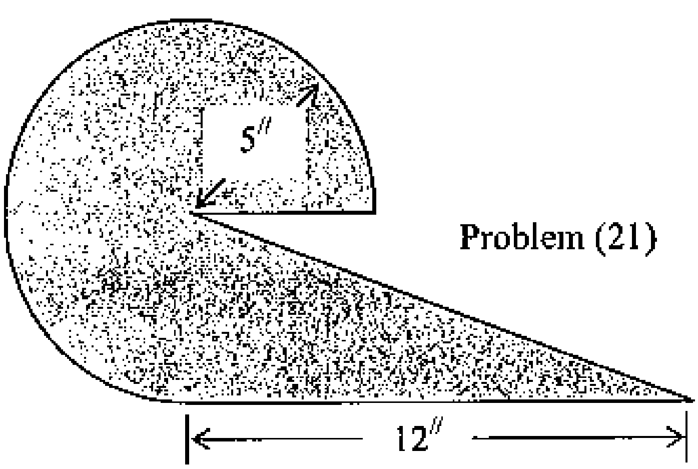
Problem (18)



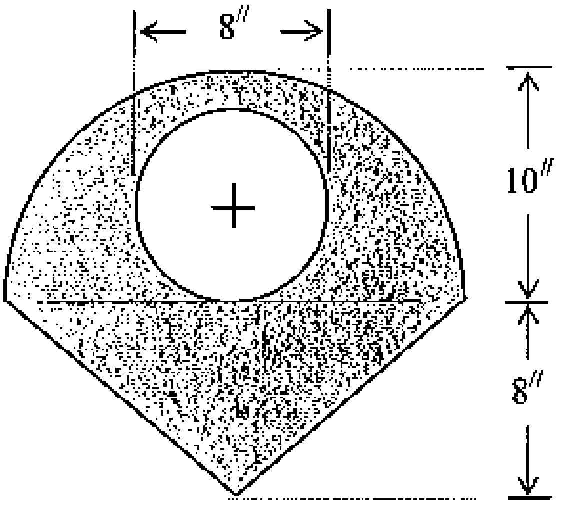
Problem (19)

Problem (20)





Problem (21)



Problem (22)

Problem (23) – (30): Determine centriod of the shaded area. The linear unit is inch.

Answer (23): $\bar{x}$ = 0.9 inch & $\bar{y}$ = 2.25 inch

Answer (24): $\bar{x}$ = 2.345 inch & $\bar{y}$ = 3.108 inch

Answer (25): $\bar{x}$ = 2.546479089 inch & $\bar{y}$ = 2.03718372 inch

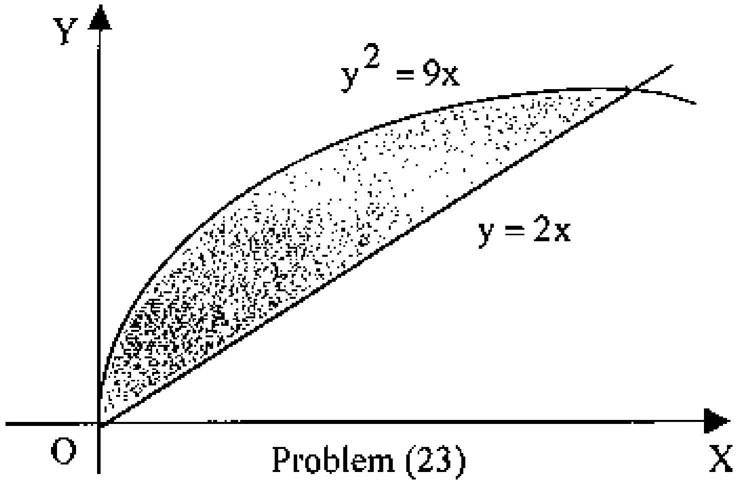
Answer (26): $\bar{x}=\frac{\pi}{2}$ & $\bar{y}=\frac{\pi}{8}$

Answer (27): $\bar{x}$ = 3 inch & $\bar{y}$ = 2.371708245 inch

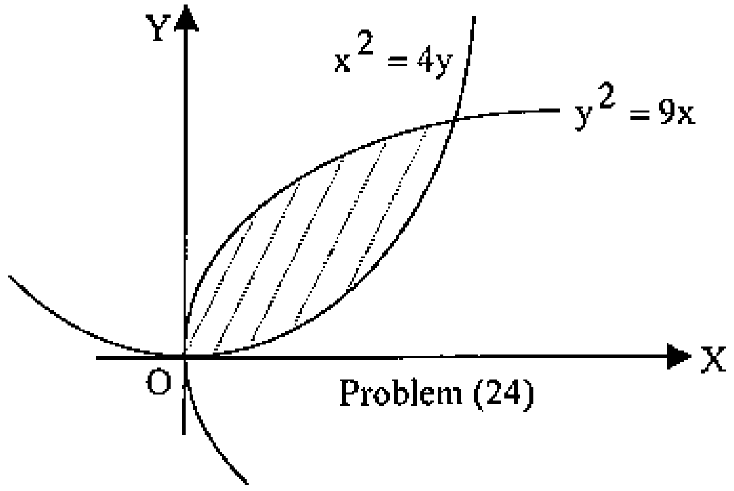
Answer (28): $\bar{x}$ = 5/6 inch & $\bar{y}$ = 3.75 inch

Answer (29): $\bar{x}$ = 6.75 inch & $\bar{y}$ = 4.05 inch

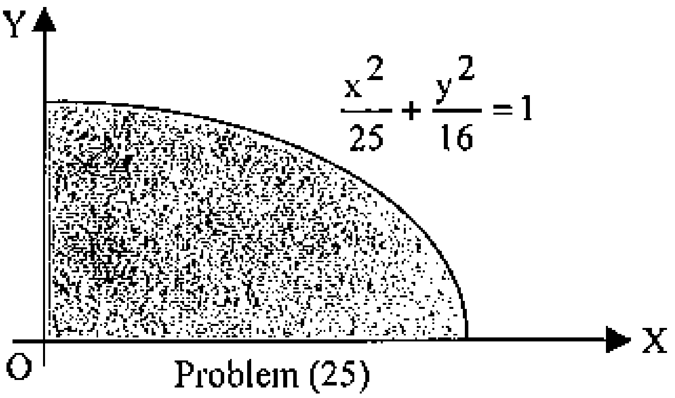
Answer (30): $\bar{x}$ = 4.20 inch & $\bar{y}$ = 0 inch



Problem (23)



Problem (24)



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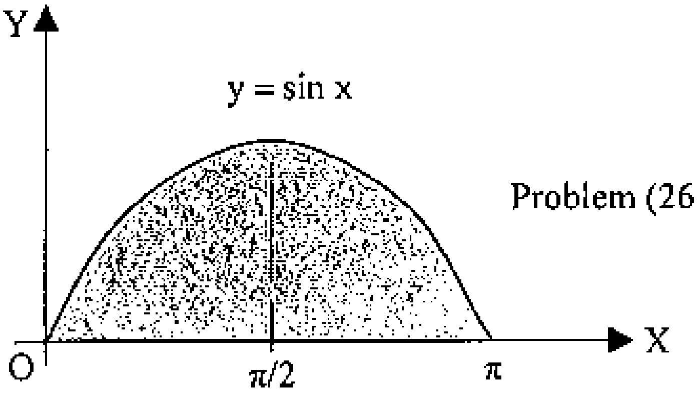
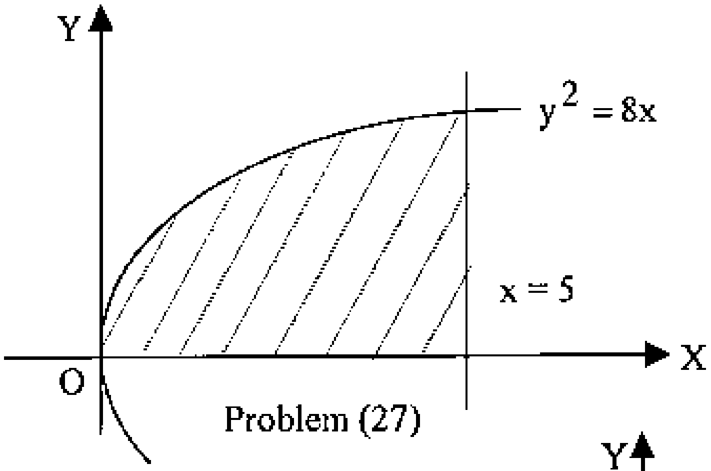
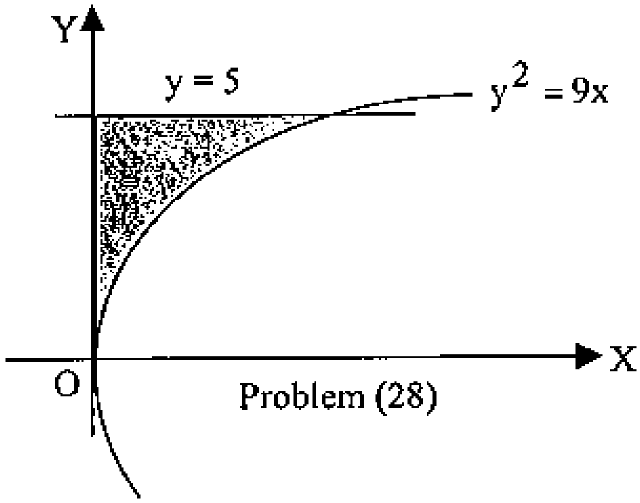


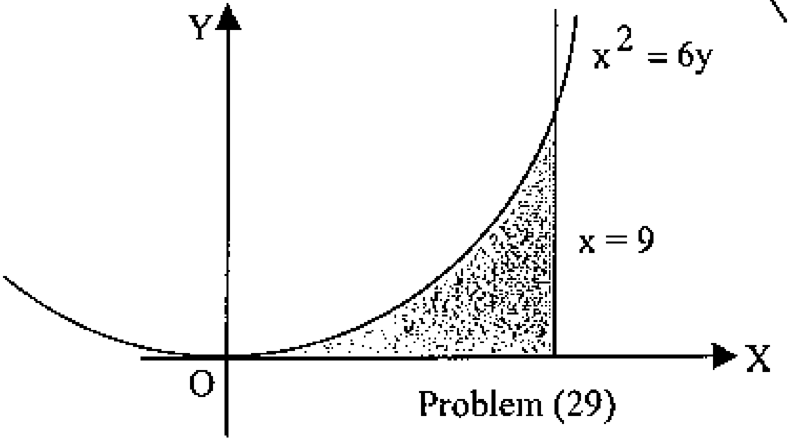
Problem (26)



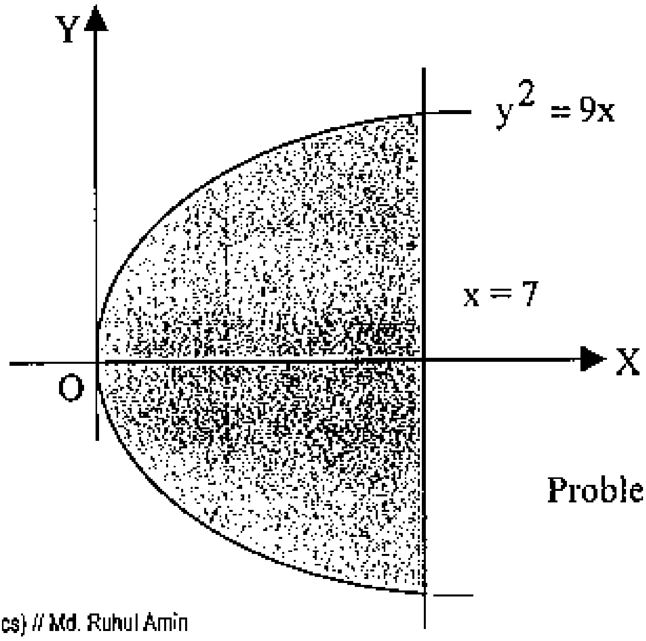
Problem (27)



Problem (28)

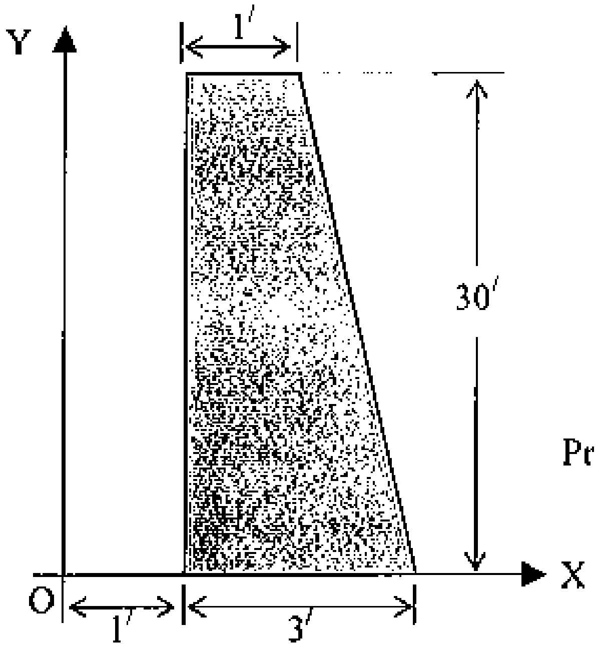


Problem (29)



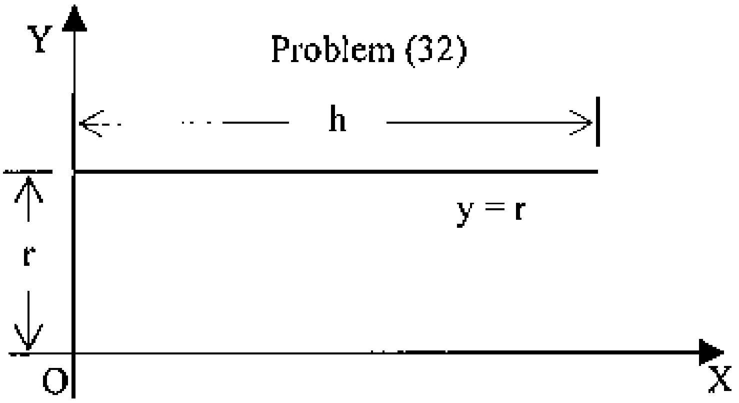
Problem (30)

Problem (31): Determine the volume generated by revolving the composite area through 360^0 about X-axis and Y-axis.
Answer (31): 4712.38899 cft & 785.3981634 cft.



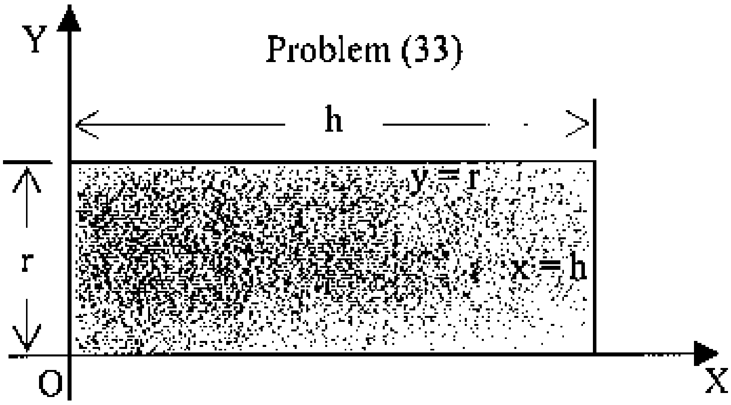
Problem (31)

Problem (32): Determine the surface area generated by revolving a straight line (parallel to X-axis) through 360^0 about X-axis.
Answer (32): $2\pi rh$



Problem (32)

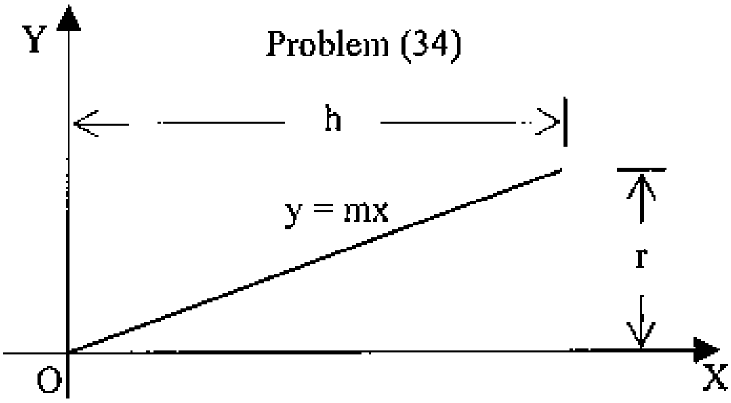
Problem (33): Determine the volume generated by revolving a rectangular area through 360^0 about X-axis.
Answer (33): $\pi r^2 h$



Problem (33)

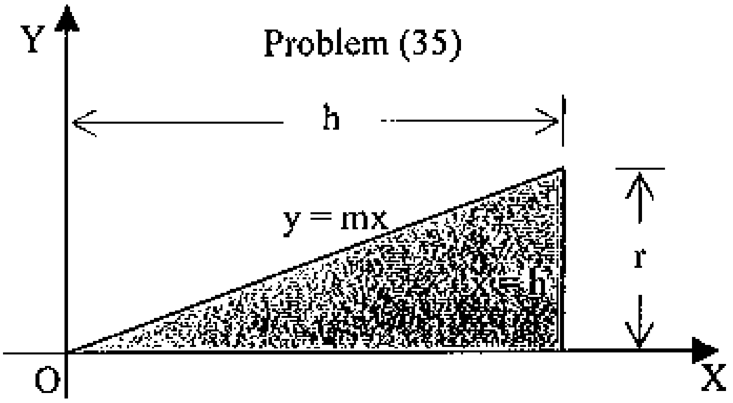
Problem (34): Determine the surface area generated by revolving an inclined straight line through 360^0 about X-axis.

Answer (34): $\pi r\sqrt{r^2 + h^2}$



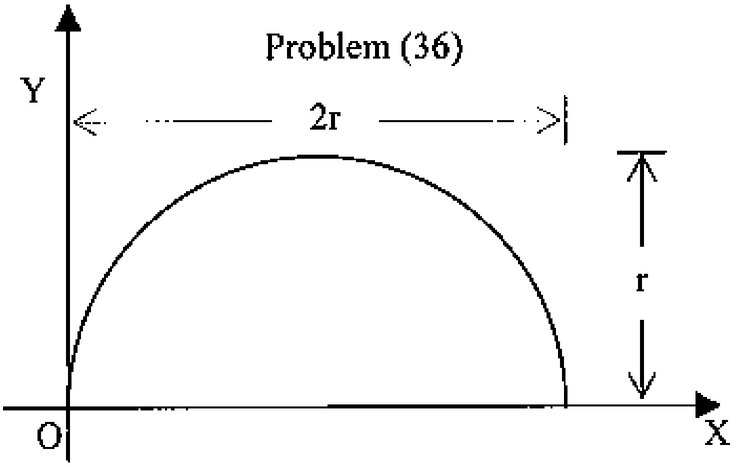
Problem (35): Determine the volume generated by revolving a triangular area through 360^0 about X-axis.

Answer (35): $\frac{1}{3}\pi r^2 h$



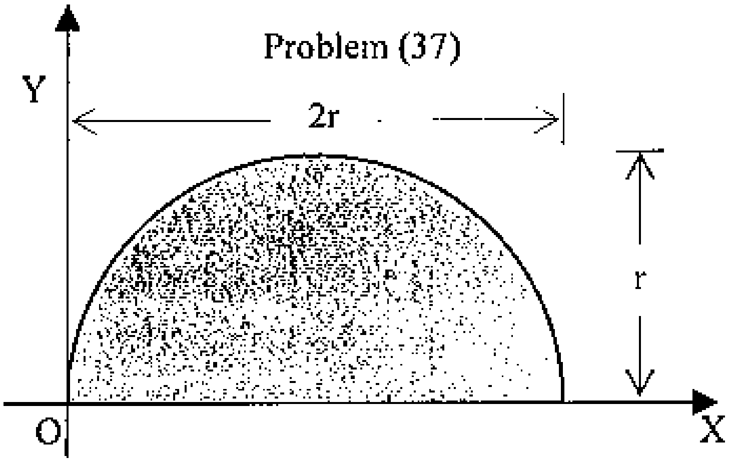
Problem (36): Determine the surface area generated by revolving a semi-circular arch through 360^0 about X-axis.

Answer (36): $4\pi r^2$



Problem (37): Determine the volume generated by revolving a semi-circular area through 360^0 about X-axis.

Answer (37): $\frac{4}{3}\pi r^3$


Problem (37)

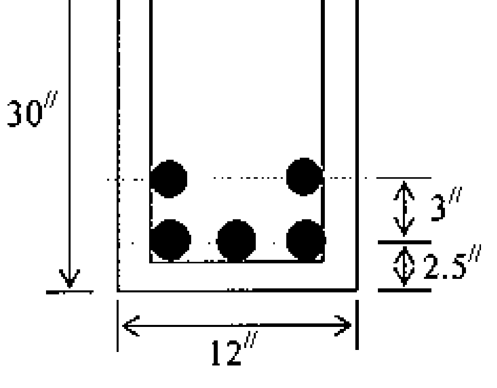
Problem (38): Prove that the centroid of a right circular cone is at a distance of ${}^{3h}/_{4}$ from its apex where h is the height of the cone.

Problem (39): Prove that the centroid of a hemi-sphere is at a distance of ${}^{3r}/_{8}$ from its base where r is the radius of the sphere.

Problem (40): Prove that the centroid of a semi-paraboloid is at a distance of ${}^{2a}/_{3}$ from its apex where a is the height of the paraboloid.

Problem (41): Prove that the centroid of a semi-ellipsoid is at a distance of ${}^{3a}/_{8}$ from its base where a is half of the distance of major axis.

Problem (42): A section of reinforced concrete beam as shown in figure contains two layer of reinforcing bars (top layer 2 – $0.75''\varphi$ & bottom layer 3 - $0.875''\varphi$). Where is the centroid of the rods?
Answer (42): 3.48525184 inch from bottom.



Problem (42)

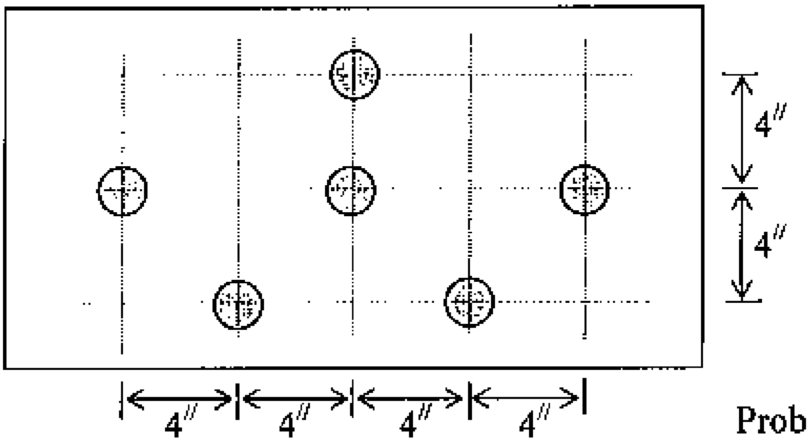
Solution (42): Top layer bars 2 – 0.75$''\varphi$ [φ indicates diameter of bar] (area of each bar = $\frac{\pi d^2}{4} = \frac{\pi(0.75)^2}{4}$ = $9\pi/64$ inch^2 & centroid located (3 + 2.5) inch = 5.5 inch from bottom) & bottom layer bars 3 - 0.875$''\varphi$ (area of each bar = $\frac{\pi d^2}{4}$ = $\frac{\pi(0.875)^2}{4}$ = $49\pi/256$ inch^2 & centroid located 2.5 inch from bottom)

$$\bar{y} = \frac{\sum Ay}{\sum A} = \frac{A_{\text{top bars}}\, y_{\text{top bars}} + A_{\text{bottom bars}}\, y_{\text{botom bars}}}{A_{\text{top bars}} + A_{\text{top bars}}}$$

$$= \frac{2*(9\pi/64)\times(5.5)+3*(49\pi/256)\times(2.5)}{2*(9\pi/64)+3*(49\pi/256)} = 3.48525184 \text{ inch from bottom.}$$

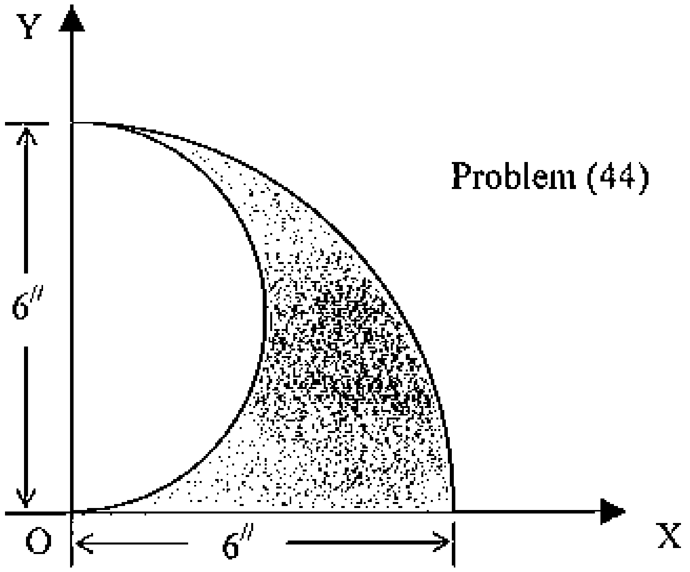
Problem (43): A gusset plate of a truss joints contains six rivets. Locate the centriod. Each rivet having a diameter of 1 inch.

Answer (43): 8 inch (from left rivet) & 3.33333333 inch (from bottom row of rivets)



Problem (43)

Problem (44): Determine the location of centroid for the shaded area.



Problem (44)

Solution (43): Quarter circle – Area (1), $A_1 = \frac{\pi(6)^2}{4} = 9\pi$ inch^2; $x_1 = \frac{4(6)}{3\pi} = \frac{8}{\pi}$ inch; $y_1 = \frac{4(6)}{3\pi} = \frac{8}{\pi}$ inch.

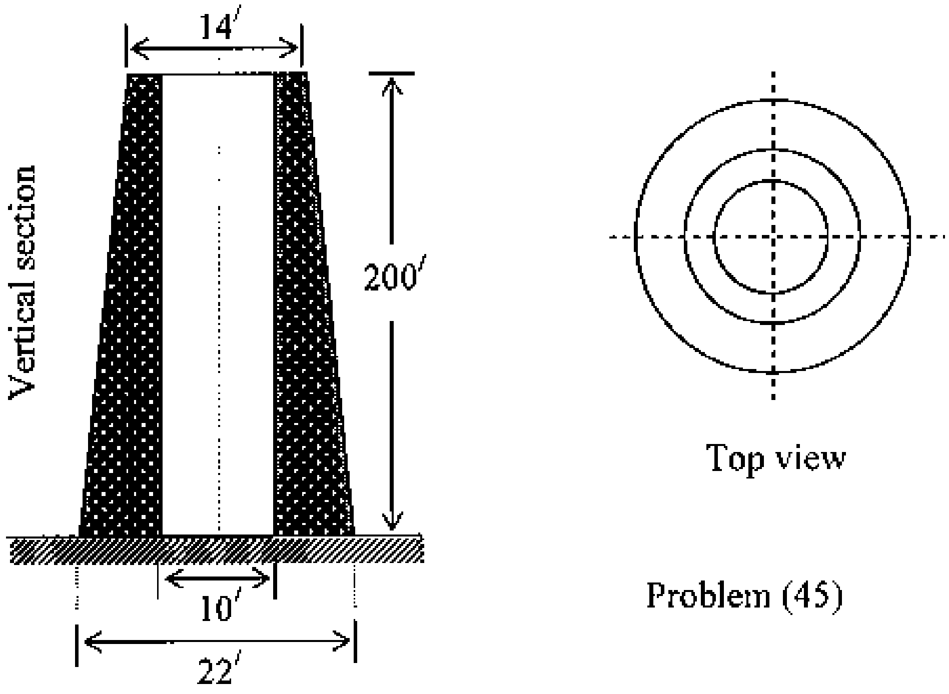
Semi circle – Area (2), $A_2 = \frac{\pi(3)^2}{2} = 4.5\pi$ inch^2; $x_2 = \frac{4(3)}{3\pi} = \frac{4}{\pi}$ inch; $y_2 = 3$ inch.

$$\bar{x} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2} = \frac{9\pi \times \frac{8}{\pi} - 4.5\pi \times \frac{4}{\pi}}{9\pi - 4.5\pi} = 3.819718634 \text{ inch}$$

$$\bar{y} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{9\pi \times \frac{8}{\pi} - 4.5\pi \times 3}{9\pi - 4.5\pi} = 2.092958179 \text{ inch}$$

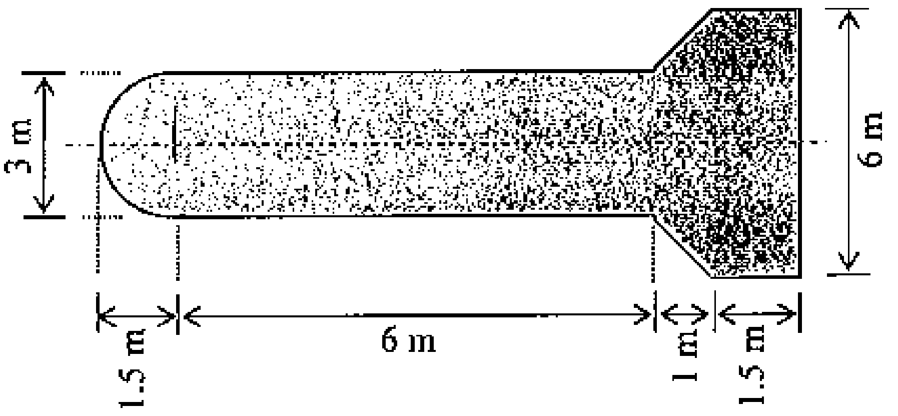
Answer (44): $\bar{x} = 3.819718634$ inch & $\bar{y} = 2.092958179$ inch

Problem (45): A section of a 200-ft smokestack is shown in figure. Consider as a solid of revolution, what volume of concrete does the stack require?
Answer (45): 36023.59576 cubic feet.



Problem (45)

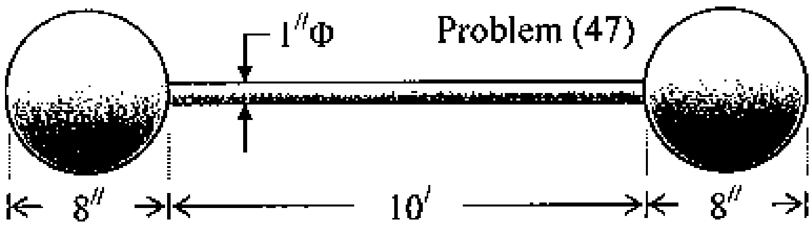
Problem (46): Considering as a volume of revolution, determine the volume generated by revolving the area about horizontal axis.
Answer (46): 171.0804856 m^3



Problem (46)

Problem (47): How much does it cost to paint the dumbbell at 1.5 tk. per square inch.

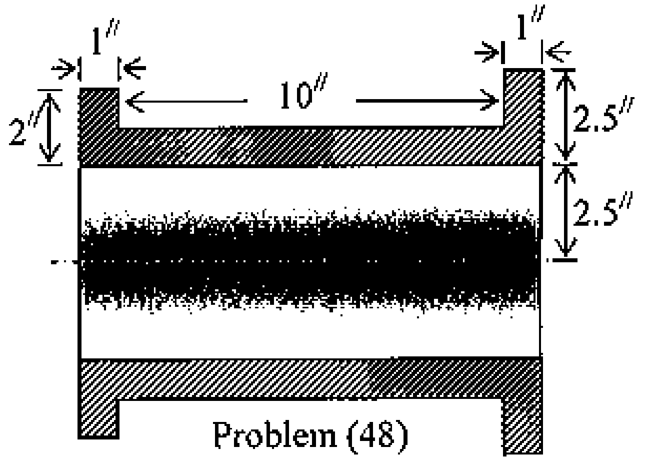
Answer (47): 1166.316273 tk.



Problem (47)

Problem (48): Determine the volume of revolution of the shaded area about horizontal axis.

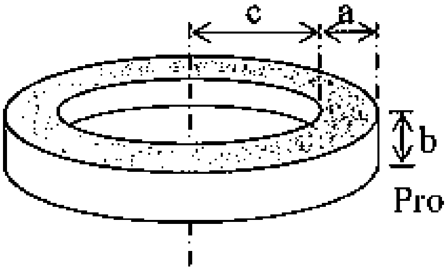
Answer (48): 291.3827186 inch^3



Problem (48)

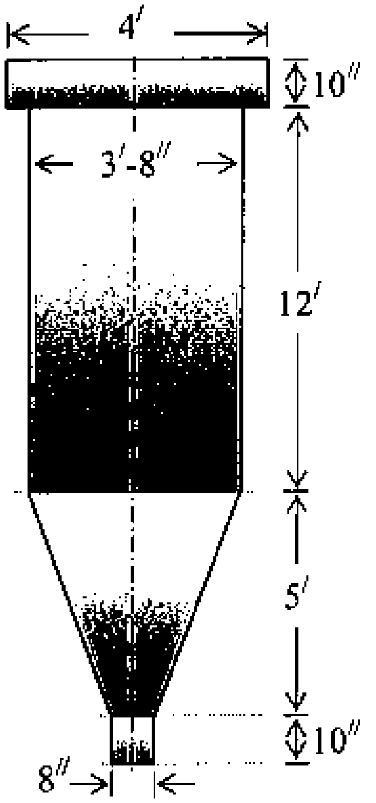
Problem (49): Determine the surface area & volume of revolution.

Answer (49): $A = 4\pi(a+b)\left(c+\frac{a}{2}\right)$; $V = 2\pi ab\left(c+\frac{a}{2}\right)$



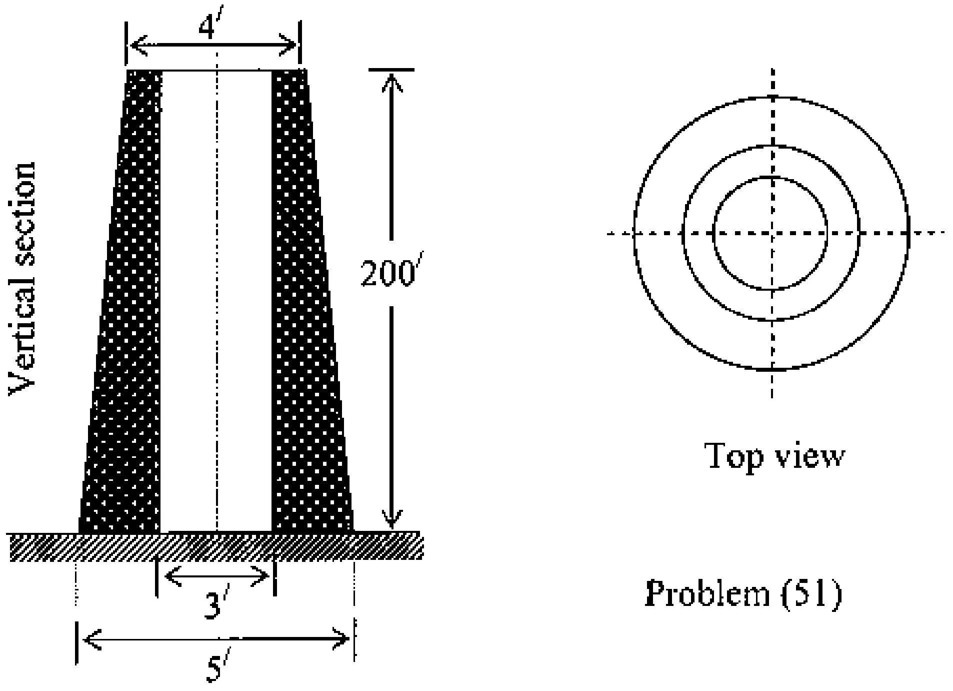
Problem (49)

Problem (50): Determine the volume of cement that can be stored in the silo.
Answer (50): 156.9632774 cft.



Problem (50)

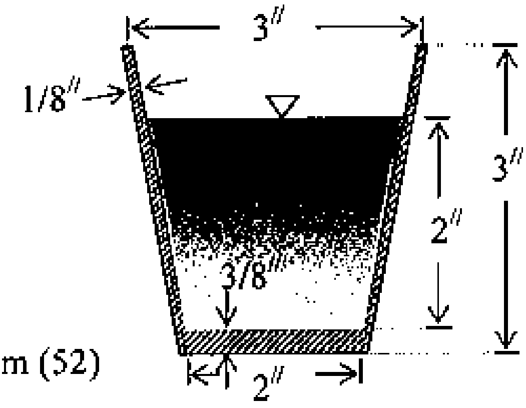
Problem (51): Determine the volume of concrete needed to construct the chimney.
Answer (51): 2565.634 cft.



Problem (51)

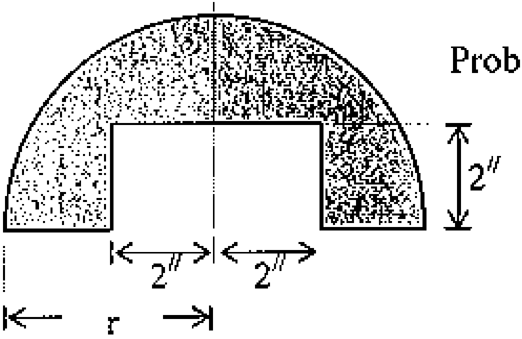
Problem (52): For the conical water glass, determine the center of gravity. (γ_{glass} = 1 oz./inch^3 & γ_{water} = 0.578 oz./inch^3)

Answer (53):



Problem (52)

Problem (53): What will be the radius of half-circle so that the centroid of the composite body will be 2.02341623 inch from bottom?



Problem (53)

Solution (53): Semi-circle – Area (1), $A_1 = \frac{\pi(r)^2}{2}$ inch^2; $y_1 = \frac{4r}{3\pi}$ inch (from bottom).

Rectangle – Area (2), A_2 = 8 inch^2; x_2 = 1 inch (from bottom).

$\bar{y}$ = 2.02341623 inch (from bottom).

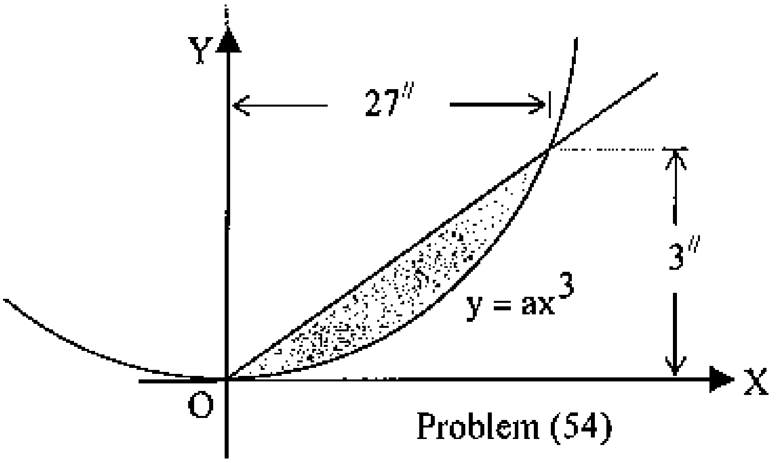
$$\bar{y} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{\frac{\pi r^2}{2} \times \frac{4r}{3\pi} - 8 \times 1}{\frac{\pi r^2}{2} - 8} = 2.02341623 \text{ inch}$$

$\Rightarrow$ r = 4 inch.

Answer (53): 4 inch.

Problem (54): Determine centroid of the shaded area.

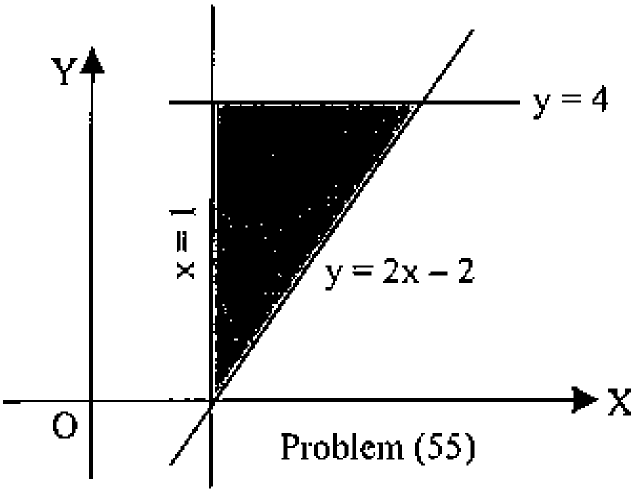
Answer (54): $\bar{x}$ =7.924058336 inch & $\bar{y}$ = 1.579522817 inch.



Problem (54)

Problem (55): Determine centroid of the shaded area bounded by three lines x = 1, y = 4 & y = 2x + 2. Use method of integration.

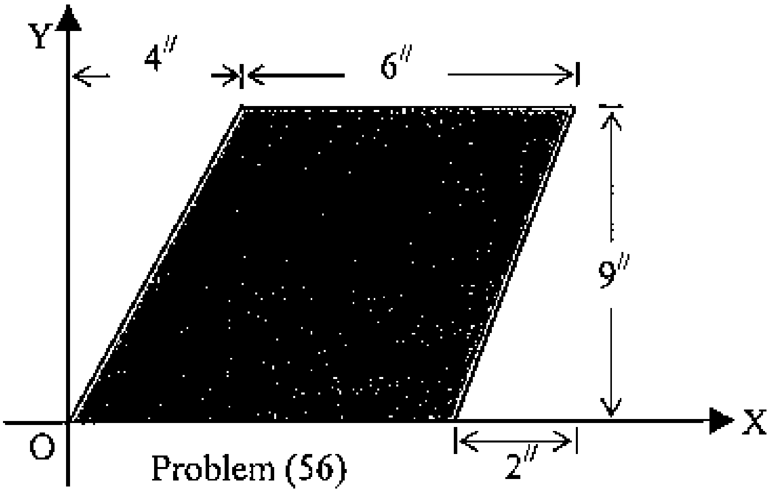
Answer (55): $\bar{x}$ = 5/3 inch & $\bar{y}$ = 8/3 inch.



Problem (55)

Problem (56): Determine centroid of the shaded area.

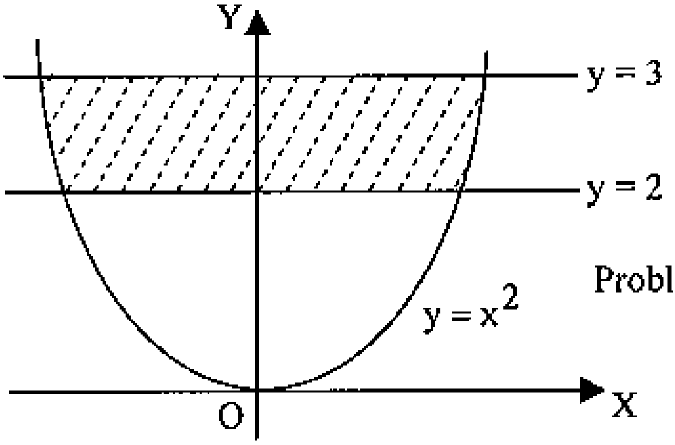
Answer (56): $\bar{x}$ = 7.769230769 inch & $\bar{y}$ = 4.615384615 inch.



Problem (56)

Problem (57): Determine centroid of the shaded area bounded by lines $y = 2$, $y = 3$ & curve $y = x^2$.

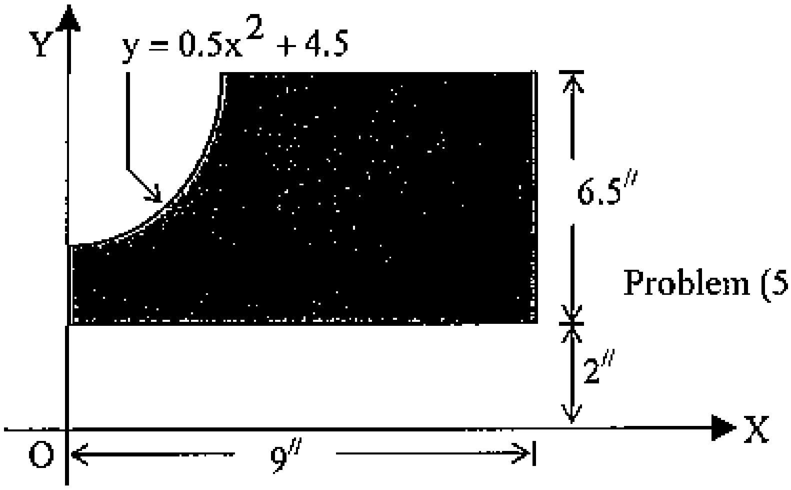
Answer (57): $\bar{x} = 0$ inch & $\bar{y} = 2.516745425$ inch.



Problem (57)

Problem (58): Determine centroid of the shaded area.

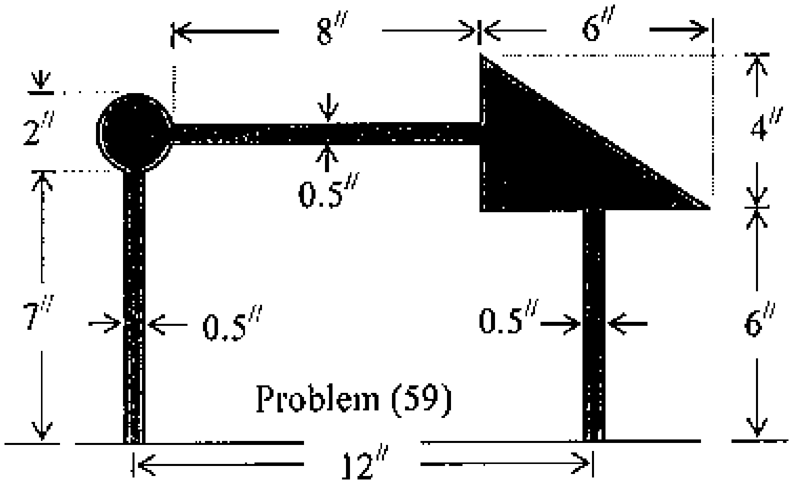
Answer (58): $\bar{x} = 3.961733897$ inch & $\bar{y} = 5.560606734$ inch.



Problem (58)

Problem (59): Determine centroid of the shaded area.

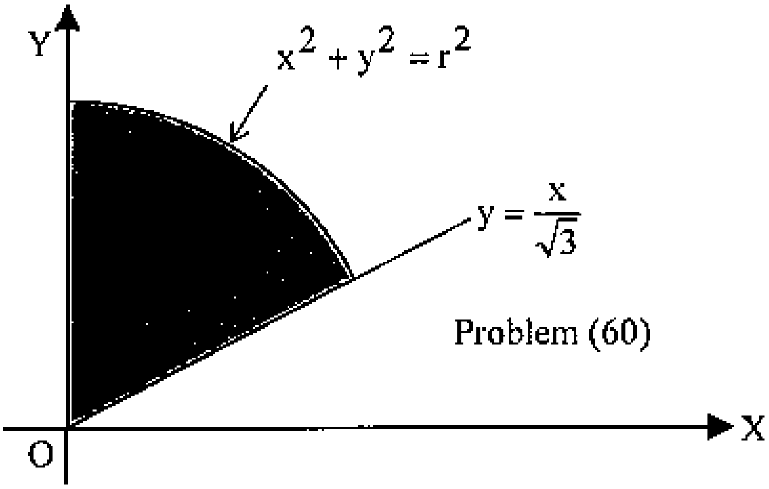
Answer (59): $\bar{x} = 7.331837867$ inch (from centerline of sphere) & $\bar{y} = 6.488783418$ inch (from bottom).



Problem (59)

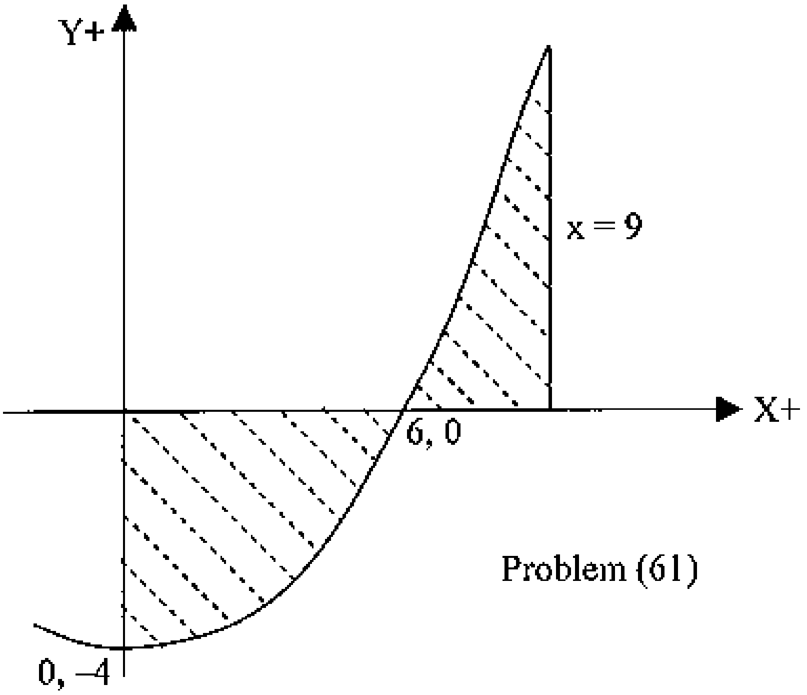
Problem (60): Determine centroid of the shaded area bounded by circle $x^2 + y^2 = r^2$ and line $y = x/\sqrt{3}$. Use method of integration.

Answer (60): $\bar{x} = \dfrac{r}{\pi}$ inch & $\bar{y} = \dfrac{r\sqrt{3}}{\pi}$ inch



Problem (61): Determine location of centroid for the second order parabola.

Answer (61): $\bar{x}$ = 4.010869565 inch & $\bar{y}$ = – 0.62173913 inch



Chapter Eight

Moment of Inertia of Areas

Moment of Inertia: Many engineering formulas, such as those relating to strength of beams, columns, and deflection of beams involve the use of a mathematical expression of the form $\int \rho^2 dA$, where ρ is the perpendicular distance from differential area dA to the axis of inertia. This integral appears so frequently that it has been named **moment of inertia***. Moment of inertia applied to areas has no real meaning when examined by itself; it is merely a mathematical expression usually denoted by the symbol I. However, when used in combination with other terms, as elastic beam formula, flexural shear stress formula, Euler's column formula, torsional shear stress formula, it begins to have significance.

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* The term moment of inertia is derived as follows: force (F) is proportional to mass (m) (i.e., inertia) of a body and its linear acceleration (a) by the equation (Newton's second law of motion) F = ma. The equation relating applied forces to the angular acceleration (α) of rotating bodies is $F.d = \left[\int \rho^2 dm\right].\alpha$. If the first equation is stated as force equal inertia times linear acceleration, then by analogy the second equation may be stated as moment of force equals moment of inertia (i.e. moment of mass) times angular acceleration. By comparison of the above statements, the expression $\left[\int \rho^2 dm\right]$ is termed as moment of inertia of mass. Similarly, for areas, the expression $\int \rho^2 dA$ is known as the moment of inertia of the area.

In translatory / linear motion, Force = mass × linear acceleration

Force = inertia × linear acceleration

$$F = ma$$

In rotating / angular motion, Moment = mass moment of inertia × angular acceleration

$$M = \left[\int \rho^2 dm\right].\alpha$$

The mathematical definition of moment of inertia, $I = \int \rho^2 dA$, indicates that the area is divided into small parts such as dA, and each area is multiplied by the square of its moment arm about the reference axis. Thus as shown in figure 1, if the co-ordinates of the centre of the differential area dA are (x, y), the moment of inertia about the X axis is the summation of the product of each area dA by the square of its moment arm y. this gives,

$$I_x = \int y^2 dA$$

Similarly, the moment of inertia about the Y axis is given by

$$I_y = \int x^2 dA$$

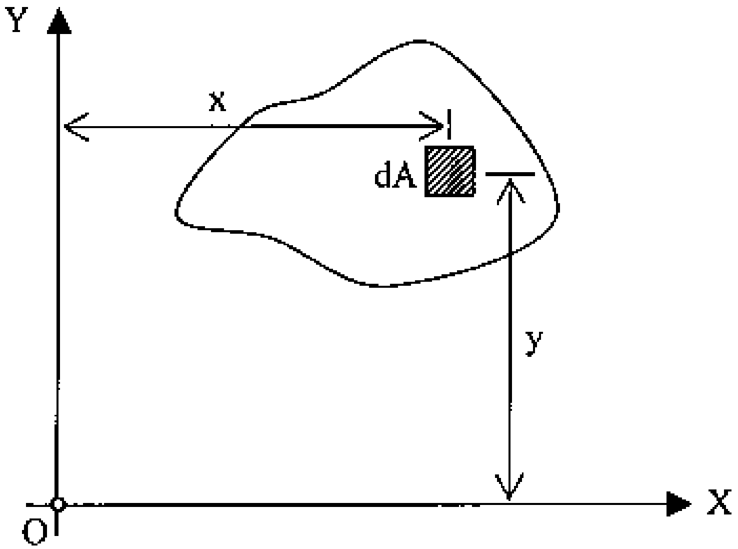


Figure 1: Definition of moment of inertia

The moment of inertia (of area) is sometimes called the **second moment of area** because each differential area multiplied by its moment arm gives the moment of area; when multiplied by a second time by its moment arm, it gives to moment of inertia. The term **second moment of area** is preferable to the expression **moment of inertia**; the latter is confusing when applied to an area having no inertia. The term moment of inertia, however, is long established and is not likely to be superseded by the other.

Examination of the integral $\int \rho^2 dA$ shows it to be a fourth dimensional term because it is composed of a distance squared multiplied by an area. So, unit of moment of inertia is $(\text{length})^4$.

Moment of inertia must always be positive.

Moment of Inertia of Masses: $I = \int \rho^2 dm$

Moment of Inertia of Area: $I = \int \rho^2 dA$

Centriod of area	Moment of inertia of area
First moment of area	Second moment of area
$\bar{x}=\frac{\int x\,dA}{\int dA}=\frac{\int x\,dA}{A}$ & $\bar{y}=\frac{\int y\,dA}{\int dA}=\frac{\int y\,dA}{A}$ $\bar{x}(A)=\int x\,dA$ & $\bar{y}(A)=\int y\,dA$	$I_x=\int y(ydA)=\int y^2dA$ & $I_y=\int x(xdA)=\int x^2dA$

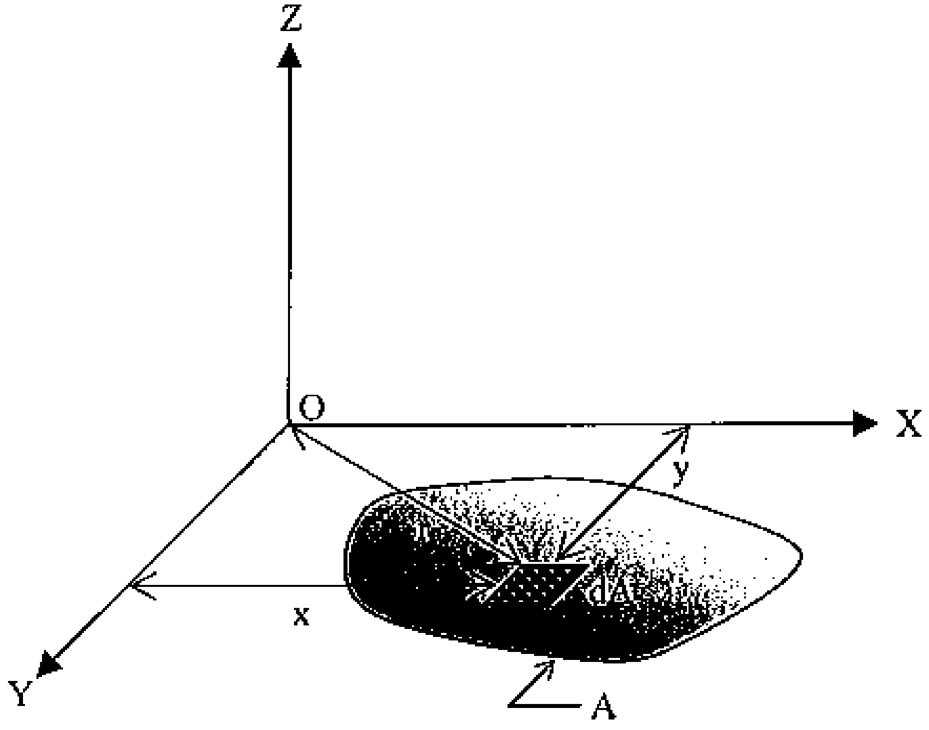


Figure 2: Rectangular and polar moment of inertia

Rectangular Moment of Inertia: Moment of inertia about an axis which lies in plane with the area is called rectangular moment of inertia. Consider the area A, in figure 2, lying in the XY plane. The differential element dA is at a distance y from the X axis & at a distance x from the Y axis.

Moment of inertia about the X axis, $I_x=\int x^2dA$

Moment of inertia about the Y axis, $I_y=\int y^2dA$

These are rectangular moments of inertia.

Polar Moment of Inertia: Moment of inertia about an axis which lies perpendicular to the plane of the area is called polar moment of inertia. The element dA, in figure 2, is at a distance r from the Z axis and since Z axis is normal to the plane of the area, the polar moment of inertia about this particular axis is

$$J = I_z = \int r^2 dA$$

Relation between Polar & Rectangular Moment of Inertia:

$$J = \int r^2 dA$$

$$r^2 = x^2 + y^2$$

$$I_z = \int r^2 dA = \int (x^2 + y^2) dA = \int (y^2 + x^2) dA$$

$$= \int y^2 dA + \int x^2 dA = I_x + I_y$$

$$J = I_z = I_x + I_y$$

This expression is an important relation which is often useful. In words, it says that the polar moment of inertia of a plane area about a particular axis perpendicular to the plane of the area is equal to the sum of two rectangular moments of inertia about axes in the plane of the area, provided that all three axes are mutually perpendicular and intersect at a common point. Since there are an infinite number of axes about which moments may be taken, there are an infinite number of moments of inertia for a particular area.

Radius of Gyration (k) (for Masses): Radius of gyration is that distance which when squared and multiplied with the total mass of the body gives the mass moment of inertia of the body. Thus, if I is the moment of inertia of a body of mass m about an axis, then its radius of gyration k about that axis is given by the relation:

$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}$$

A physical meaning may be assigned to this term. Radius of gyration is the distance at which the entire mass can be assumed to be concentrated such that the moment of the actual body and the concentrated mass is the same.

Radius of Gyration (k) (for Areas): Radius of gyration 'k' is a mathematical term defined by the relation

$$k = \sqrt{\frac{I}{A}}$$

Where, I is the moment of inertia of the given section and A is the area of the section.

Suffixes with moment of inertia term accompany with the radius of gyration also. Thus, we have

$$k_{\text{(about the same axis about which moment of inertia is taken)}} = \sqrt{\frac{I_{\text{(about an axis)}}}{A}}$$

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

From the above the definition of radius of gyration, we can note down the following relation:

$$I = Ak^2$$

Looking at the above equation, we can assign a meaning to the term radius of gyrations.

Radius of gyration of a given area an axis is the distance of the area at which, if it is squeezed and placed, there is no change in the moment of inertia (see figure 3).

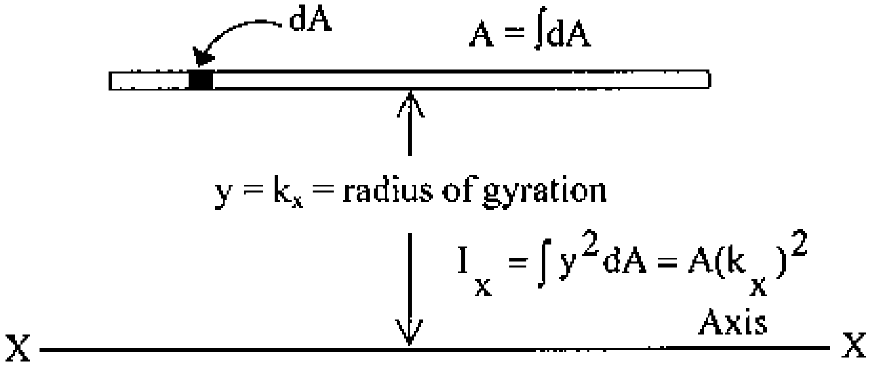

dA
$A = \int dA$
$y = k_x$ = radius of gyration
$I_x = \int y^2 dA = A(k_x)^2$
Axis
X
X

Figure 3(a): Definition of radius of gyration

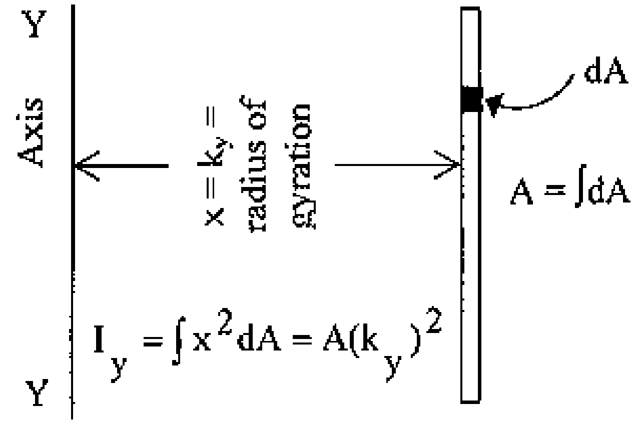

Y
Axis
Y
$x = k_y$ = radius of gyration
dA
$A = \int dA$
$I_y = \int x^2 dA = A(k_y)^2$

Figure 3(b): Definition of radius of gyration

Theorem of Moment of Inertia: The following two theorems of moment of inertia are useful in handling many engineering problems:

(1) Perpendicular Axis Theorem
(2) Parallel Axis Theorem
(3) Rotation Axes Theorem

Transfer Formula - Perpendicular Axis Theorem: Moment of inertia of a given area about an axis perpendicular to the plane (polar moment of inertia) through a point 'O' is equal to the sum of moment of inertia of that area about any two perpendicular axes through that point, the axes being in the plane of the area.

$$I_z = I_x + I_y$$

Transfer Formula - Parallel Axis Theorem: Moment of inertia of an area about any axis in the plane of the area is equal to the sum of moment of inertia of that area about the centroidal parallel axis of the area and the product of area and the square distance of the centroid of the area from the axis.

$$(I)_{\text{about any axis}} = (I)_{\text{about a parallel axis at centroid}} + Ad^2$$

$\Rightarrow\!> I = \bar{I} + Ad^2$

Where, I = any axis about which moment of inertia is needed (i.e., moment of inertia of an area with respect to any non-centroidal axis)

$\bar{I}$ = moment of inertia of the area about centoidal axis (parallel to the axis about which moment of inertia is needed)
A = area
d = perpendicular distance between centroidal axis and reference axis.

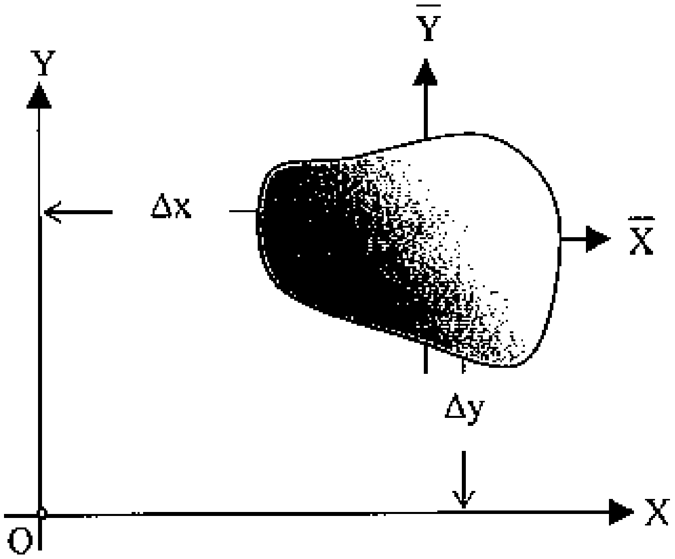


Figure 4: Parallel axis theorem

$$I_x = \overline{I_x} + A\,(\Delta y)^2$$

$$I_y = \overline{I_y} + A\,(\Delta x)^2$$

Adding these two equation, $I_x + I_y = \overline{I_x} + A\,(\Delta y)^2 + \overline{I_y} + A\,(\Delta x)^2$

$$(I_x + I_y) = (\overline{I_x} + \overline{I_y}) + A\,(\Delta x + \Delta y)^2$$

$$J = \bar{J} + A(\Delta r)^2$$

Now, $I = \bar{I} + Ad^2$

$Ak^2 = A\bar{k}^2 + Ad^2$

$k = \bar{k} + d^2$

Δx = (x centroid of one area of the composite body – x centroid of whole composite area) = $(x_1 - \bar{x})$, $(x_2 - \bar{x})$, $(x_3 - \bar{x})$, ………… $(x_i - \bar{x})$.

Δy = (y centroid of one area of the composite body – y centroid of whole composite area) = $(y_1 - \bar{y})$, $(y_2 - \bar{y})$, $(y_3 - \bar{y})$, ………… $(y_i - \bar{y})$.

Transfer Formula - Rotation Axes Theorem: See article "Moment of Inertia about Inclined Axes".

Moment of Inertia from First principle (i.e. by integration):

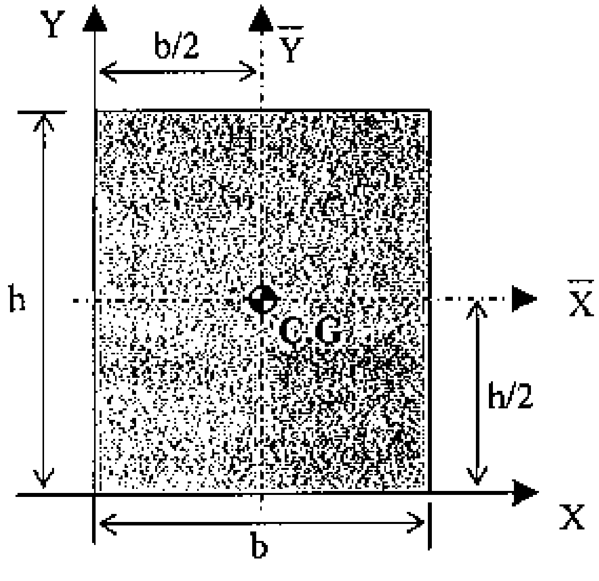


Figure 5(a): Rectangle

X & Y = Bases of rectangle

$\bar{X}$ & $\bar{Y}$ = Centroidal axes of rectangle

- Moment of Inertia of a Rectangle (about Centroidal Axis)

$$\bar{I}_x = \frac{b(h)^3}{12} \text{ and } \bar{I}_y = \frac{h(b)^3}{12}$$

- Moment of Inertia of a Rectangle (about its Base)

$$I_x = \frac{b(h)^3}{3} \text{ and } I_y = \frac{h(b)^3}{3}$$

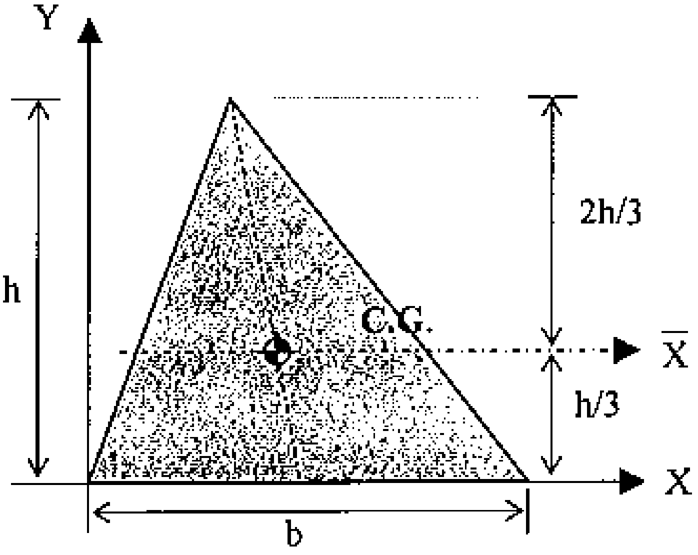


Figure 5(b): Triangle

X = Base of triangle

$\bar{X}$ = Centroidal axis of triangle

- Moment of Inertia of a Trinagle (about Centroidal Axis)

$$\bar{I}_x = \frac{b(h)^3}{36}$$

- Moment of Inertia of a Triangle (about its Base)

$$I_x = \frac{b(h)^3}{12}$$

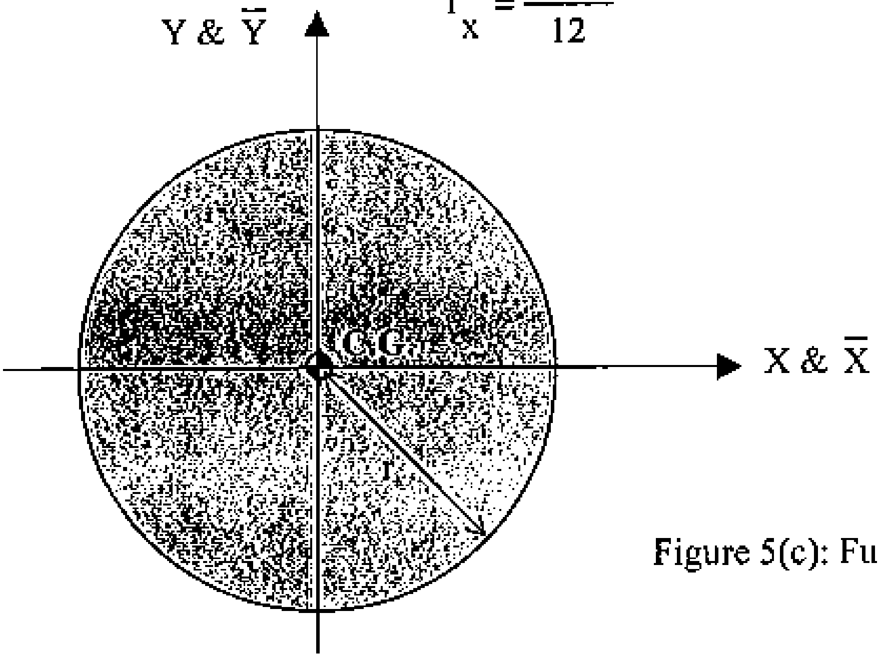


Figure 5(c): Full circle

X & Y = Diametral axes of full-circle
$\overline{X}$ & $\overline{Y}$ = Centroidal axes of full-circle
▶ Moment of Inertia of a Circle (about Diametral & Centroidal Axis)

$$I_x = \overline{I_x} = \frac{\pi(r)^4}{4}$$

$$I_y = \overline{I_y} = \frac{\pi(r)^4}{4}$$

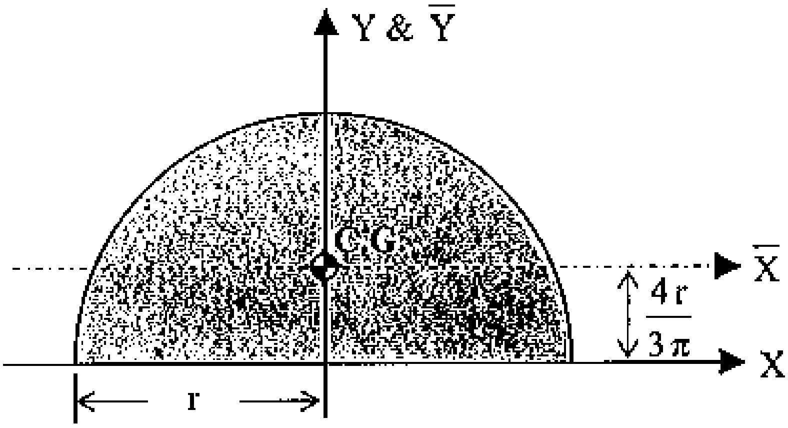


Figure 5(d): Semi-circle

X & Y = Diametral axes of semi-circle
$\overline{X}$ & $\overline{Y}$ = Centroidal axes of semi-circle
▶ Moment of Inertia of a Semi-Circle (about Diametral & Centroidal Axis)

$$I_x = \frac{\pi(r)^4}{8}$$

$$\overline{I_x} = I_x - A(\Delta y)^2$$

$$\overline{I_x} = \frac{\pi(r)^4}{8} - \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right)^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4$$

$$\overline{I_x} = 0.10975696(r)^4$$

$$I_y = \frac{\pi(r)^4}{8}$$

$$\overline{I_y} = I_y - A(\Delta x)^2$$

$$\overline{I_y} = \frac{\pi(r)^4}{8} - \left(\frac{\pi r^2}{2}\right)(0)^2$$

$$\overline{I}_y = \frac{\pi(r)^4}{8}$$

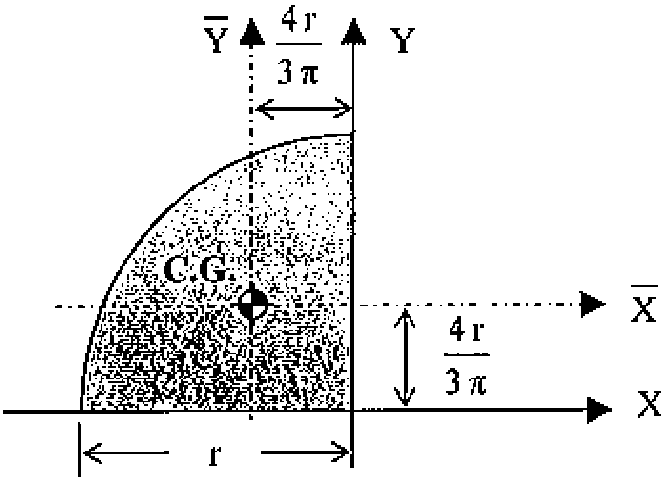


Figure 5(e): Quarter-circle

X & Y = Diametral axes of quarter-circle
$\overline{X}$ & $\overline{Y}$ = Centroidal axes of quarter-circle
► Moment of Inertia of a Quarter-Circle (about Diametral & Centroidal Axis)

$$I_x = \frac{\pi(r)^4}{16}$$

$$\overline{I}_x = I_x - A(\Delta y)^2$$

$$\overline{I}_x = \frac{\pi(r)^4}{16} - \left(\frac{\pi r^4}{4}\right)\left(\frac{4r}{3\pi}\right)^2 = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)r^4$$

$$\overline{I}_x = 0.05487848(r)^4$$

$$I_y = \frac{\pi(r)^4}{16}$$

$$\overline{I}_y = I_y - A(\Delta x)^2$$

$$\overline{I}_y = \frac{\pi(r)^4}{16} - \left(\frac{\pi r^4}{4}\right)\left(\frac{4r}{3\pi}\right)^2 = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)r^4$$

$$\overline{I}_y = 0.05487848(r)^4$$

Moment of Inertia from Second Principle (i.e. by summation) / Moment of Inertia of Composite Section: Moment of inertia of composite sections about an axis can be determined by following the steps given below:

(1) Divide the composite section into a suitable manner of simple figures.
(2) Locate the centroid of each simple figure.
(3) Using standard expressions, determine the moment of inertia of each simple figure about its centroid. To each of these terms, add product of area and square of distance of its centroid from the axis under consideration (parallel axis theorem) to get moment of inertia of each simple figure about the axis under consideration.
(4) Sum up moment of inertia of all simple figures about the axis under consideration to get the moment of inertia of the composite section.

Product of Inertia: The product of inertia is a mathematical expression of the form $\int xy\,dA$ and is denoted by the symbol P_{xy}. The product of inertia relates an area directly to a set of axes by the following formulation.

$$P_{xy} = \int_A xy\,dA$$

The product of inertia is not used as much often as the moment of inertia but is needed in such problems as determining maximum and minimum moments of inertia, unsymmetrical bending of beams, and structural analysis of indeterminate frames.

The unit of product of inertia is of the same form as that of the moment of inertia, namely$(\text{length})^4$.

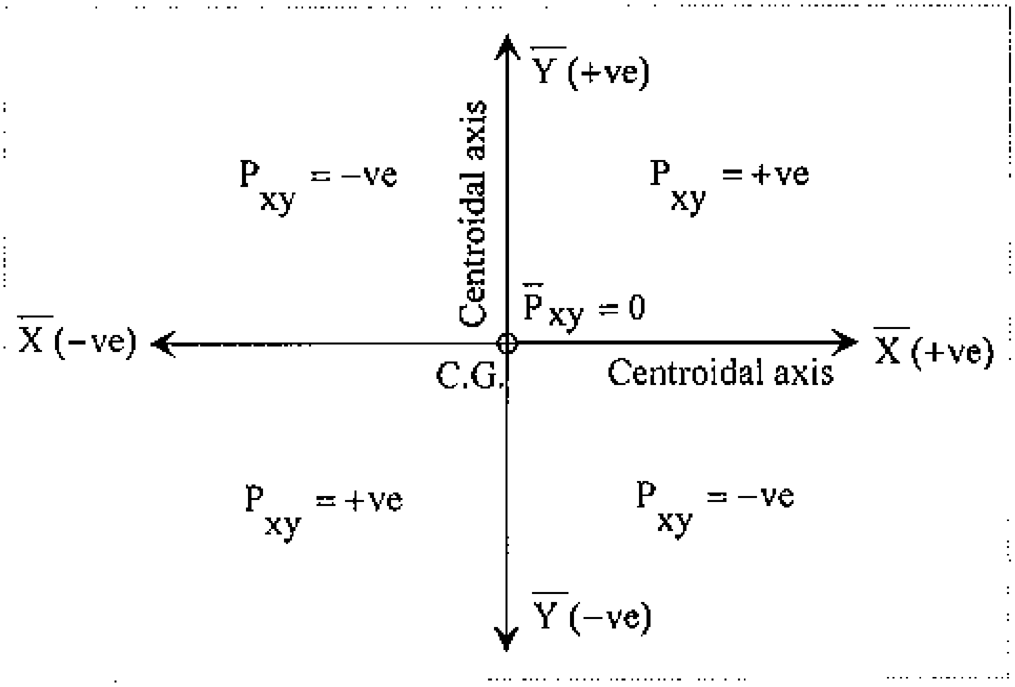


Figure 6: Sign of product of inertia (P_{xy})

In the expression $P_{xy} = \int xy\, dA$, each elementary area dA is multiplied by the product of the co-ordinates of the area; hence product of inertia $\left(P_{xy}\right)$ may be positive, negative or zero. If the whole area lies in the first quadrant, the product of inertia will be positive since both x and y are positive for each elementary area. Similarly, if the entire area lies in the third quadrant, both x and y will be negative and hence product of inertia will be positive. However, if the entire area lies either in the second quadrant or in the fourth quadrant, product of inertia will be negative.

If one axis is the axis of symmetry, the product of inertia of the area with respect to the pair of axes is zero. Thus, the product of inertia with respect to the centroidal axes is zero.

Transfer Formula for Product of Inertia – Parallel Axes: See figure 4

$$\left(P_{xy}\right)_{\text{about any axis}} = \left(P_{xy}\right)_{\text{about a parallel axis at centroid}} + A(\Delta x)(\Delta y)$$

$$P_{xy} = \overline{P_{xy}} + A(\Delta x)(\Delta y)$$

Where, P_{xy} = any set of axes about which product of inertia is needed

$\overline{P_{xy}}$ = product of inertia about centroidal axes

A = area

Δx = distance between X-axis and $\overline{X}$-axis = (x centroid of one area of the composite body – x centroid of whole composite area) = $(x_1 - \overline{x})$, $(x_2 - \overline{x})$, $(x_3 - \overline{x})$, ... $(x_i - \overline{x})$.

Δy = distance between Y-axis and $\overline{Y}$-axis = (y centroid of one area of the composite body – y centroid of whole composite area) = $(y_1 - \overline{y})$, $(y_2 - \overline{y})$, $(y_3 - \overline{y})$, ... $(y_i - \overline{y})$.

Moment of Inertia about Inclined Axes: In some cases, it is necessary to determine the moment of inertia with respect to axes which are inclined to the usual axes. The moment of inertia in such cases can be obtained by formal integration, but a general formula is easier to use.

The problem may be stated as follows: Assuming the values of I_x, I_y and P_{xy}, with respect to the X and Y axes to be known, determine the values of

$I_{x'}$, $I_{y'}$ and $P_{x'y'}$, with respect to the X' and Y' axes inclined at angle θ with the X and Y axes, as shown in figure 7.

The coordinate for a typical differential area dA are given by x and y with respect to the X and Y axes, and by x' and y' relative to the X' and Y' axes. The relations between these coordinates can be obtained by projecting x and y the X' and Y' axes. This gives

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$I'_x = \int \left(y'\right)^2 dA = \int (-x\sin\theta + y\cos\theta)^2 dA$$

$$I'_y = \int \left(x'\right)^2 dA = \int (x\cos\theta + y\sin\theta)^2 dA$$

$$P_{x'y'} = \int x'y'dA = \int (x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta)dA$$

$$I'_x = \int (y^2\cos^2\theta - 2xy\sin\theta xys\theta + x^2\sin^2\theta)dA$$

$$I'_y = \int (y^2\sin^2\theta + 2xy\sin\theta xys\theta + x^2\cos^2\theta)dA$$

$$P_{x'y'} = \int (y^2 \sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta - x^2\sin\theta\cos\theta)dA$$

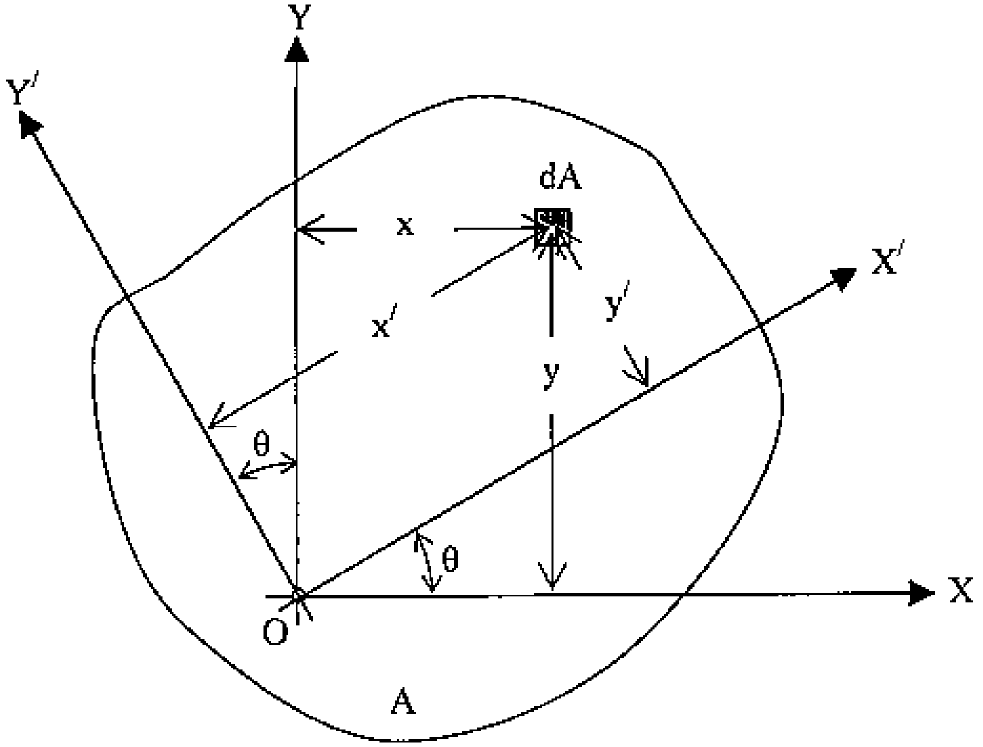


Figure 7: Moment of inertia about inclined axes

$$I_{x'} = I_x \cos^2\theta - P_{xy}\sin 2\theta + I_y \sin^2\theta$$

$$I_{y'} = I_x \sin^2\theta + P_{xy}\sin 2\theta + I_y \cos^2\theta$$

$$P_{x'y'} = \frac{I_x}{2}\sin 2\theta + P_{xy}\cos^2\theta - P_{xy}\sin^2\theta - \frac{I_y}{2}\sin 2\theta$$

If the relations $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ &

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - P_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + P_{xy}\sin 2\theta$$

$$P_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + P_{xy}\cos 2\theta$$

Anticlockwise θ is taken as positive & clockwise θ is taken as negative in the above formulae.

$$I_x + I_y = I_{x'} + I_{y'} = I_{x''} + I_{y''} = I_{x'''} + I_{y'''} = \ldots\ldots\ldots = \text{constant} = I_z$$

This shows that the sum of the moments of inertia with respect to any set of rectangular axes through the same point is a constant quantity and is equal to the polar moment of inertia about an axis passing through the same point.

Maximum and Minimum Moment of Inertia & Principal Axes: In accordance with the calculus, the $\frac{dI_{x'}}{d\theta}$ set equal to zero will yield the value of θ for which $I_{x'}$ is a maximum or minimum. Differentiating the equation of $I_{x'}$ with respect to θ, equating the derivatives to zero, we have

$$I_{x'} = I_x \cos^2\theta - P_{xy}\sin 2\theta + I_y \sin^2\theta$$

$$\Rightarrow \frac{dI_{x'}}{d\theta} = -I_x(\sin 2\theta) - 2P_{xy}(\cos 2\theta) + I_y(\sin 2\theta) = 0$$

$$\Rightarrow \tan 2\theta = \frac{2P_{xy}}{I_y - I_x}$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2P_{xy}}{I_y - I_x}\right),$$ the angles defining the maximum & minimum moment of inertia.

We find two values of θ differing by 90^0 (i.e., θ and $90^0+\theta$), and thus, we have two axes defined. These axes are called **principal axes** for the origin in question. About one of these axes, the moment of inertia is a maximum; about the other, a minimum & the product moment of inertia is zero.

When the above value of θ is put in the expression of $I_{x^/}$ and $I_{y^/}$, these will give the maximum and minimum moment of inertia ($I_{maximum}$ & $I_{minimum}$).

Alternatively, you can use the following expression

$$I_{max} = \left(\frac{I_x + I_y}{2}\right) + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$I_{min} = \left(\frac{I_x + I_y}{2}\right) - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$I_x + I_y = I_{x^/} + I_{y^/} = I_{max} + I_{min} = \text{constant} = I_z$$

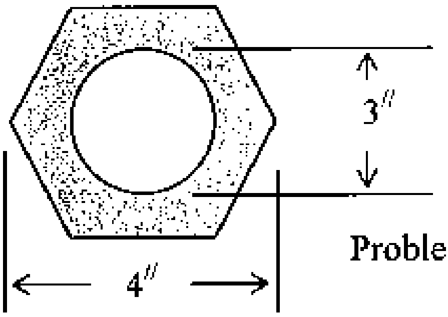
$$k_{max} = \sqrt{\left(\frac{I_{max}}{A}\right)}$$

$$k_{min} = \sqrt{\left(\frac{I_{min}}{A}\right)}$$

Problems

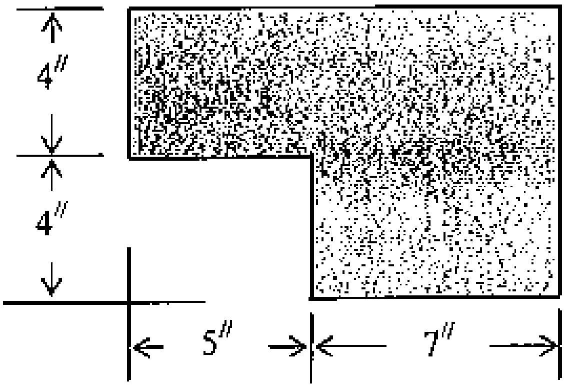
Problem (1): Determine minimum radius of gyration about centroidal axis.

Answer (1): $k_{min} = 0.671368028$ inch



Problem (1)

Problem (2): Determine minimum radius of gyration about centroidal axis.



Problem (2)

Solution (2): Whole rectangle – (1); $A_1 = 12'' \times 8'' = 96$ inch^2; $x_1 = 6''$; $y_1 = 4''$

Small rectangle – (2); $A_2 = 5'' \times 4'' = 20$ inch^2; $x_2 = 2.5''$; $y_2 = 2''$

$$\bar{x} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2} = \frac{96 \times 6 - 20 \times 2.5}{96 - 20} = 6.921052632 \text{ inch}$$

$$\bar{y} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{96 \times 4 - 20 \times 2}{96 - 20} = 4.526315789 \text{ inch}$$

$$I_{\bar{x}} = I_{x1} - I_{x2} = \left[\overline{I_{x1}} + A_1 (y_1 - \bar{y})^2\right] - \left[\overline{I_{x2}} + A_2 (y_2 - \bar{y})^2\right]$$

$$I_{\bar{x}} = \left[\frac{12(8)^3}{12} + 96(4.526315789 - 4)^2\right] - \left[\frac{5(4)^3}{12} + 20(4.526315789 - 2)^2\right]$$

$$I_{\bar{x}} = 384.2807018 \text{ inch}^4$$

$$I_{\bar{y}} = I_{y1} - I_{y2} = \left[\overline{I_{y1}} + A_1 (x_1 - \bar{x})^2\right] - \left[\overline{I_{y2}} + A_2 (x_2 - \bar{x})^2\right]$$

$$I_{\bar{y}} = \left[\frac{8(12)^3}{12} + 96(6.921052632 - 6)^2\right] - \left[\frac{4(5)^3}{12} + 20(6.921052632 - 2.5)^2\right]$$

$I_{\bar{y}} = 800.8596491 \text{ inch}^4$

$P_{\overline{xy}} = P_{xy1} - P_{xy2}$

$$P_{\overline{xy}} = P_{xy1} - P_{xy2} = \left[P_{\overline{xy1}} + A_1(x_1 - \bar{x})(y_1 - \bar{y})\right] - \left[P_{\overline{xy2}} + A_2(x_2 - \bar{x})(y_2 - \bar{y})\right]$$

$P_{\overline{xy}} = [96(6 - 6.691052632)(4 - 4.4526315789]$

$-[20(2.5 - 6.691052632)(2 - 4.526315789)]$

$P_{\overline{xy}} = (+34.91634348) - (211.758448) = -176.8421045 \text{ inch}^4$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2P_{xy}}{I_y - I_x}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2 \times (-176.8421045)}{800.8596491 - 384.2807018}\right)$$

$\theta = -20.16597549^{O}$ (clockwise)

$I_{x'} = I_x\cos^2\theta - P_{xy}\sin 2\theta + I_y\sin^2\theta$

$=(384.2807018)\times[\cos(-20.16597549^0)]^2-(-176.8421045)\times[\sin\{2\times(-20.16597549^0)\}] + (800.8596491)\times[\sin(-20.16597549^0)]^2$

$= 319.3346823 \text{ inch}^4 = I_{\text{minimum}}$

$I_{y'} = I_x\sin^2\theta + P_{xy}\sin 2\theta + I_y\cos^2\theta$

$=(384.2807018)\times[\sin(-20.16597549^0)]^2+(-176.8421045)\times[\sin\{2\times(-0.16597549^0)\}]+ (800.8596491)\times[\cos(-20.16597549^0)]^2$

$= 865.805667 \text{ inch}^4 = I_{\text{maximum}}$

Check: $I_{\bar{x}} + I_{\bar{y}} = I_{x'} + I_{y'}$; Ok

$A = A_1 - A_2 = 96 - 20 = 76 \text{ inch}^2$

$$k_{\text{minimum}} = \sqrt{\left(\frac{I_{\text{minimum}}}{A}\right)} = \sqrt{\frac{319.3346823}{76}} = 2.049822464 \text{ inch.}$$

Answer (2): $k_{\text{min}} = 2.049822464$ inch

Note: Alternatively, you can use the following expression of I_{max} & I_{min}

$$I_{max} = \left(\frac{I_x + I_y}{2}\right) + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$= \left(\frac{384.2807018 + 800.8596491}{2}\right)$$

$$+ \sqrt{\left(\frac{384.2807018 - 800.8596491}{2}\right)^2 + (-176.842105)^2}$$

$$= 592.5701755 + 273.2354932 = 865.805667 \text{ inch}^4$$

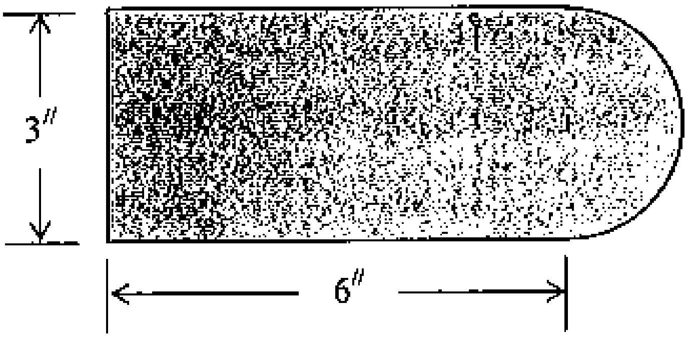
$$I_{min} = \left(\frac{I_x + I_y}{2}\right) - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$= \left(\frac{384.2807018 + 800.8596491}{2}\right)$$

$$- \sqrt{\left(\frac{384.2807018 - 800.8596491}{2}\right)^2 + (-176.842105)^2}$$

$$= 592.5701755 - 273.2354932 = 319.3346823 \text{ inch}^4$$

Problem (3): Determine minimum radius of gyration about centroidal axis.
Answer (3): $k_{min} = 0.885169079$ inch

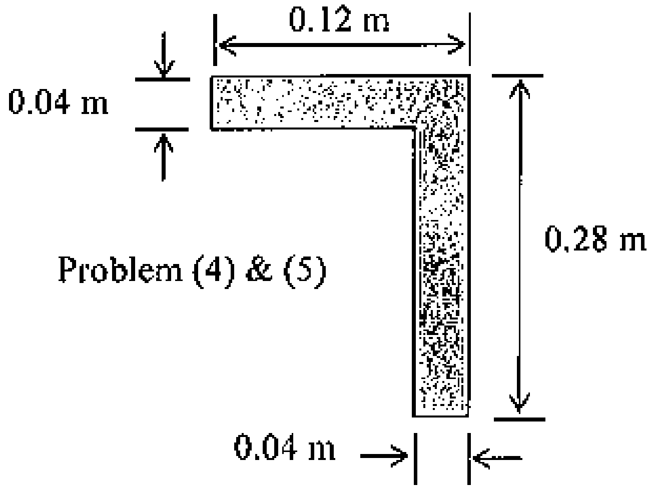


Problem (3)

Problem (4): For the angle section, determine (a) moment of inertias about horizontal and vertical centriodal axis & product of inertia about these set of axes; (b) maximum and minimum moment of inertias & its inclination; (c) minimum radius of gyration.

(a) $I_{\bar{x}} = 10.994 \times 10^{-5}\ m^4$, $I_{\bar{y}} = 1.216 \times 10^{-6}\ m^4$ & $P_{xy} = -1.792 \times 10^{-5}\ m^4$;

(b) $I_{max} = 11.264 \times 10^{-5}\ m^4$, $I_{min} = 0.986 \times 10^{-5}\ m^4$ & $\theta = +10.1^0$ (counterclockwise) with respect to horizontal centroidal axis; (c) k_{min} = 0.024944 m.



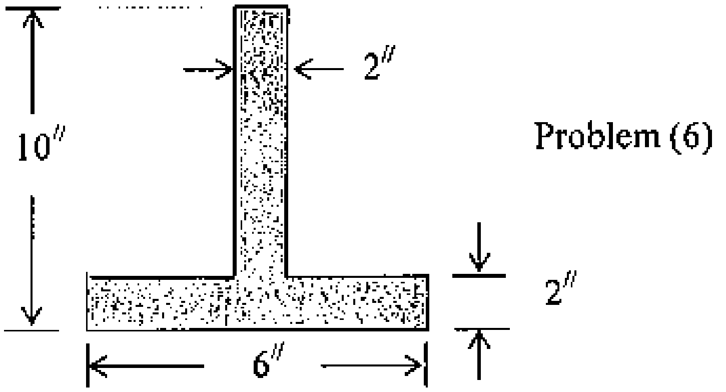
Problem (4) & (5)

Problem (5): Determine moment of inertias about an axis which is inclined at 30^0 (clockwise) with respect to horizontal centroidal axis. Also determine the product of inertia with respect to these inclined axes.

Answer (5): $I_{x'} = 6.962 \times 10^{-5}\ m^4$, $I_{y'} = 5.198 \times 10^{-5}\ m^4$ & $P_{x'y'} = -5.991 \times 10^{-5}\ m^4$.

Problem (6): Determine moment of inertias & radius of gyrations with respect to centriodal axis for the inverted T-section.

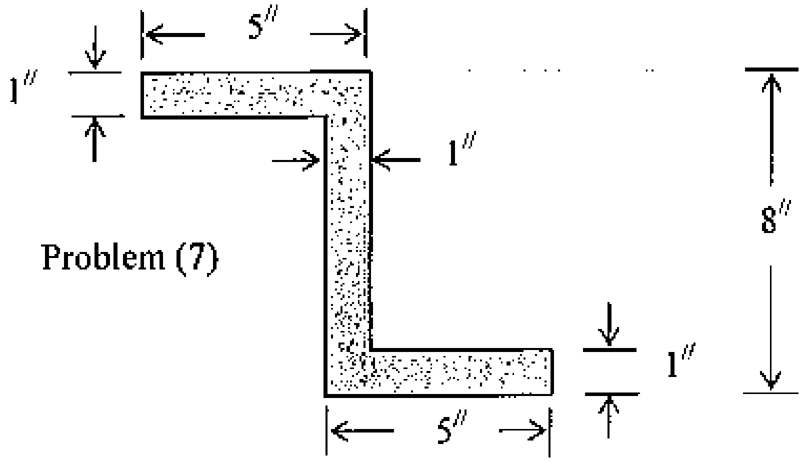
Answer (6): (a) $I_{\bar{x}} = 260.77$ inch^4, $I_{\bar{y}} = 41.33$ inch^4, $k_{\bar{x}} = 3.05$ inch & $k_{\bar{y}} = 1.21$ inch.



Problem (6)

Problem (7): Determine minimum radius of gyration for the Z-section.

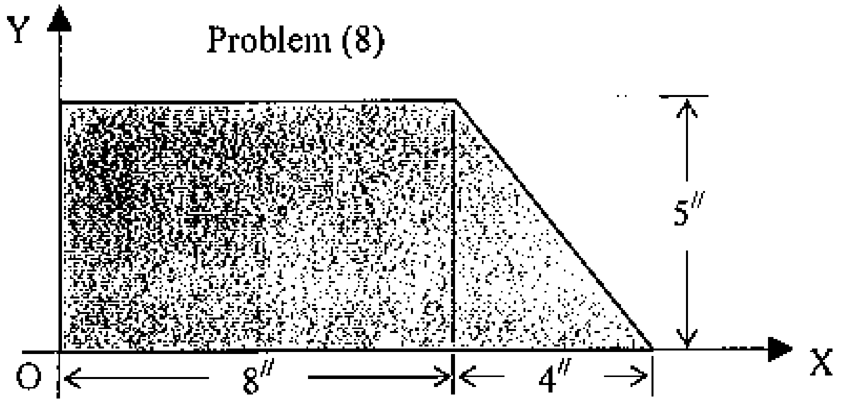
Answer (7): $k_{min} = 1.136$ inch.



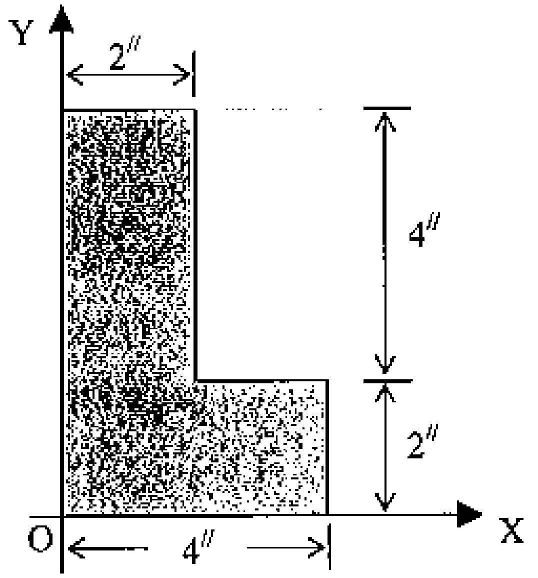
Problem (7)

Problem (8): Determine product of inertia with respect to XY axes.
Answer (8): $P_{xy} = +558$ inch^4

Problem (9): Determine product of inertia with respect to XY axes.
Answer (9): $P_{xy} = +48$ inch^4

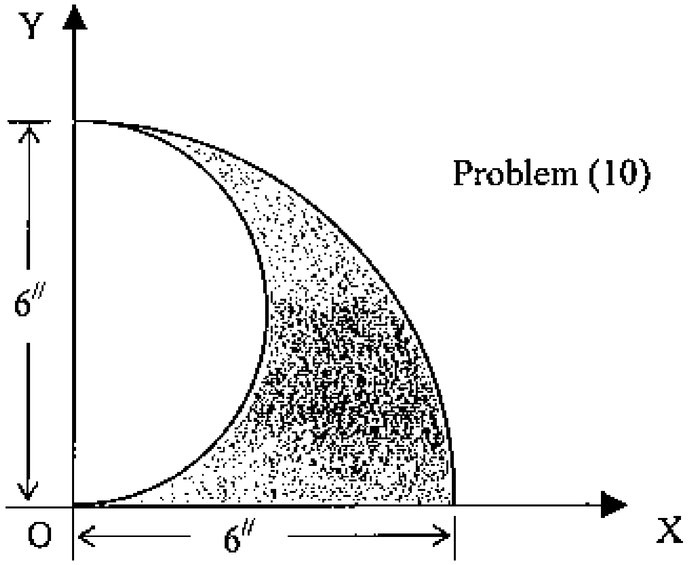


Problem (8)



Problem (9)

Problem (10): For the shaded area (bounded by a quarter-circle & a semi-circle), determine (a) moment of inertias about horizontal and vertical centriodal axis & product of inertia about these set of axes; (b) maximum and minimum moment of inertias & its inclination; (c) minimum radius of gyration.



Problem (10)

Solution (10): Quarter circle – Area (1), $A_1 = \dfrac{\pi(6)^2}{4} = 9\pi\ \text{inch}^2$; $x_1 = \dfrac{4(6)}{3\pi} = \dfrac{8}{\pi}$ inch; $y_1 = \dfrac{4(6)}{3\pi} = \dfrac{8}{\pi}$ inch.

Semi circle – Area (2), $A_2 = \dfrac{\pi(3)^2}{2} = 4.5\pi\ \text{inch}^2$; $x_2 = \dfrac{4(3)}{3\pi} = \dfrac{4}{\pi}$ inch; $y_2 = 3$ inch.

$$\bar{x} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2} = \frac{9\pi \times \frac{8}{\pi} - 4.5\pi \times \frac{4}{\pi}}{9\pi - 4.5\pi} = 3.819718634\ \text{inch}$$

$$\bar{y} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2} = \frac{9\pi \times \frac{8}{\pi} - 4.5\pi \times 3}{9\pi - 4.5\pi} = 2.092958179\ \text{inch}$$

$$I_{\bar{x}} = I_{x1} - I_{x2} = \left[\overline{I_{x1}} + A_1(y_1 - \bar{y})^2\right] - \left[\overline{I_{x2}} + A_2(y_2 - \bar{y})^2\right]$$

$$I_{\bar{x}} = \left[0.05487848(6)^4 + 9\pi(2.992958179 - \frac{8}{\pi})^2\right]$$

$$-\left[\frac{\pi(3)^4}{8} + 4.5\pi(2.092958179 - 3)^2\right] = 33.49838508\ \text{inch}^4$$

$$I_{\bar{y}} = I_{y1} - I_{y2} = \left[\overline{I_{y1}} + A_1(x_1 - \bar{x})^2\right] - \left[\overline{I_{y2}} + A_2(x_2 - \bar{x})^2\right]$$

$$I_{\bar{y}} = \left[0.05487848(6)^4 + 9\pi(3.819718634 - \frac{8}{\pi})^2\right]$$

$$-\left[0.10975696(3)^4 + 4.5\pi(3.819718634 - \frac{4}{\pi})^2\right] = 16.3955271\,\text{inch}^4$$

$$P_{\bar{xy}} = P_{xy1} - P_{xy2}$$

$$P_{\bar{xy}} = P_{xy1} - P_{xy2} = \left[P_{\bar{xy1}} + A_1(x_1 - \bar{x})(y_1 - \bar{y})\right] - \left[P_{\bar{xy2}} + A_2(x_2 - \bar{x})(y_2 - \bar{y})\right]$$

$$P_{\bar{xy}} = \left[9\pi(\frac{8}{\pi} - 3.819718634)(\frac{8}{\pi} - 2.092958179)\right]$$

$$-\left[4.5\pi(\frac{8}{\pi} - 3.819718634)(3 - 2.092958179)\right] = +16.3275279\,\text{inch}^4$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2P_{xy}}{I_y - I_x}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2 \times (+16.32675279)}{16.32675279 - 33.49838508}\right)$$

$\theta = -31.17800307^{\circ}$ (clockwise)

$I_{x'} = I_x \cos^2\theta - P_{xy}\sin 2\theta + I_y \sin^2\theta$

$=(33.49838508)\times[\cos(-31.17800307^0)^2] - (+16.32675279)\times[\sin\{2\times(31.17800307^0\}] + (16.32675279)\times[\sin(-31.17800307^0)^2]$

$= 43.35922434\ \text{inch}^4 = I_{\text{maximum}}$

$I_{y'} = I_x \sin^2\theta + P_{xy}\sin 2\theta + I_y \cos^2\theta$

$=(33.49838508)\times[\sin(-31.17800307^0)^2] + (+16.32675279)\times[\sin\{2\times(31.17800307^0\}] + (16.32675279)\times[\cos(-31.17800307^0)^2]$

$= 6.465913535\ \text{inch}^4 = I_{\text{minimum}}$

Check: $I_{\bar{x}} + I_{\bar{y}} = I_{x'} + I_{y'}$; Ok

$A = A_1 - A_2 = 9\pi - 4.5\pi = 4.5\pi\ \text{inch}^2$

$$k_{\text{minimum}} = \sqrt{\left(\frac{I_{\text{minimum}}}{A}\right)} = \sqrt{\frac{6.645913535}{4.5\pi}} = 0.676291225\ \text{inch.}$$

Note: Alternatively, you can use the following expression of I_{max} & I_{min}

$$I_{max} = \left(\frac{I_x + I_y}{2}\right) + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$= \left(\frac{33.49838508 + 16.39557271}{2}\right)$$

$$+ \sqrt{\left(\frac{33.49838508 + 16.39557271}{2}\right)^2 + (+16.39557271)^2}$$

$$= 24.91256894 + 18.4466554 = 43.35922434 \text{ inch}^4$$

$$I_{min} = \left(\frac{I_x + I_y}{2}\right) - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$= \left(\frac{33.49838508 + 16.39557271}{2}\right)$$

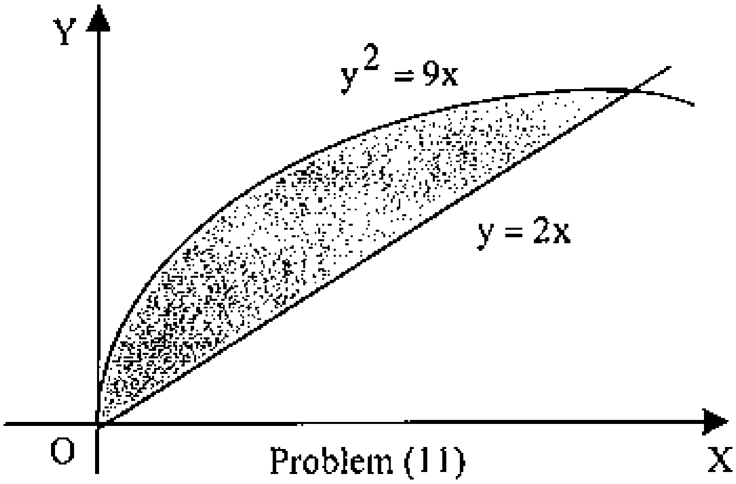
$$- \sqrt{\left(\frac{33.49838508 + 16.39557271}{2}\right)^2 + (+16.39557271)^2}$$

$$= 24.91256894 - 18.4466554 = 6.465913535 \text{ inch}^4$$

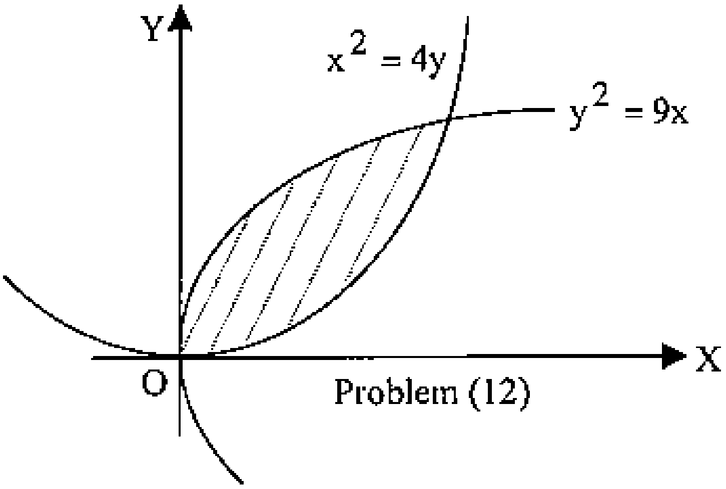
Answer (10): (a) $I_{\bar{x}}$ = 33.49838508 inch^4, $I_{\bar{y}}$ = 16.39557271 inch^4 & P_{xy} = +16.39557271 inch^4; (b) I_{max} = 43.35922434 inch^4, I_{min} = 6.465913535 inch^4 & θ = +31.17800307^0 (clockwise) with respect to horizontal centroidal axis; (c) k_{min} = 0.676291225 inch.

Problem (11) – (17): Determine moments of inertias with respect to X & Y axes and product of inertia with respect to theses set of axes. (Linear unit is in inch).
Answer (11): I_x = 10.2515625 inch^4; I_y = 1.830636161 inch^4; P_{xy} = + 4.271484375 inch^4. Answer (12): I_x = 145.5634649 inch^4; I_y = 84.77426619 inch^4; P_{xy} = + 109 inch^4. Answer (13): I_x = 169.6548085 inch^4; I_y = 225.8769757 inch^4; P_{xy} = + 165.666666 inch^4. Answer (14): I_x = 69.444444 inch^4; I_y = 5.103207264 inch^4; P_{xy} = + 16.07510289 inch^4. Answer (15): I_x = 1054.446429 inch^4; I_y = 1968.30 inch^4; P_{xy} = + 1230.1875 inch^4. Answer (16):

$I_x = \frac{5\pi}{48}$ inch^4; $I_y = \frac{\pi^3}{6} - \frac{\pi}{4}$ inch^4; $P_{xy} = + \frac{3\pi^2}{32}$ inch^4. Answer (17): $I_x = 32$ inch^4; $I_y = 12$ inch^4; $P_{xy} = + \frac{56}{3}$ inch^4.



Problem (11)



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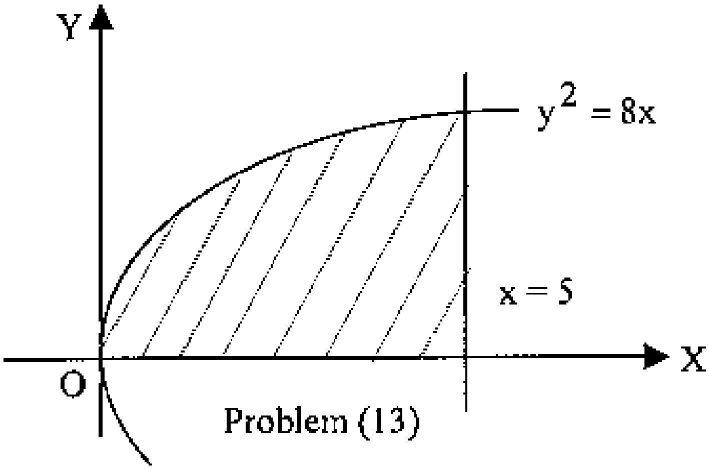

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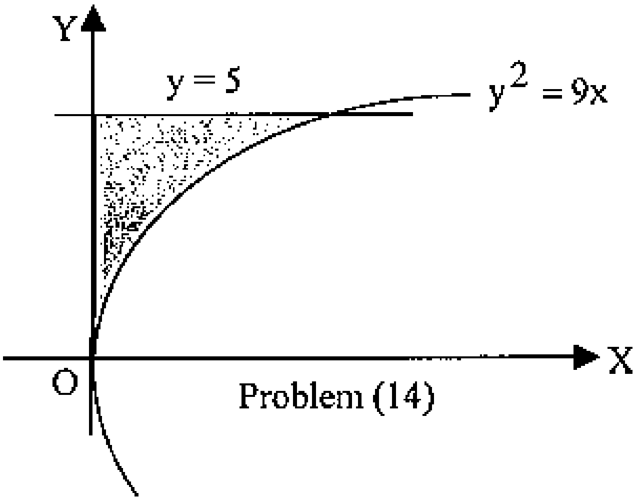

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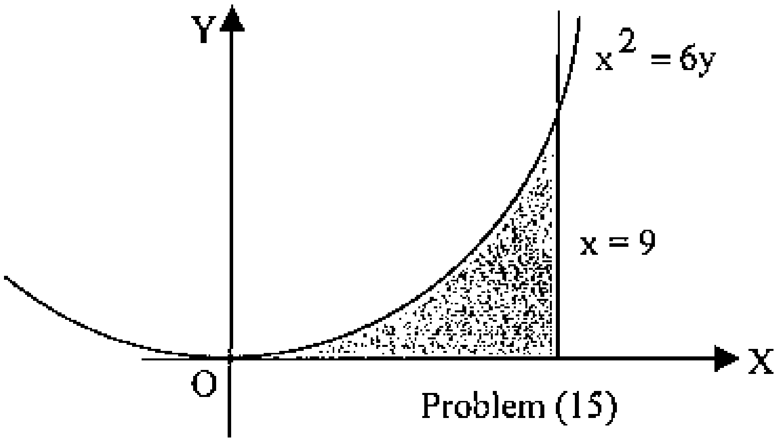

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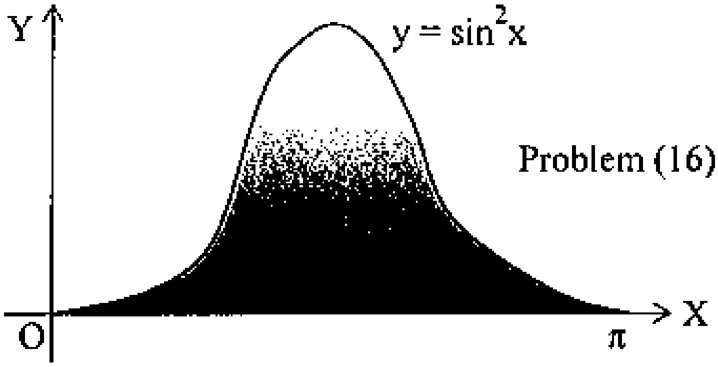
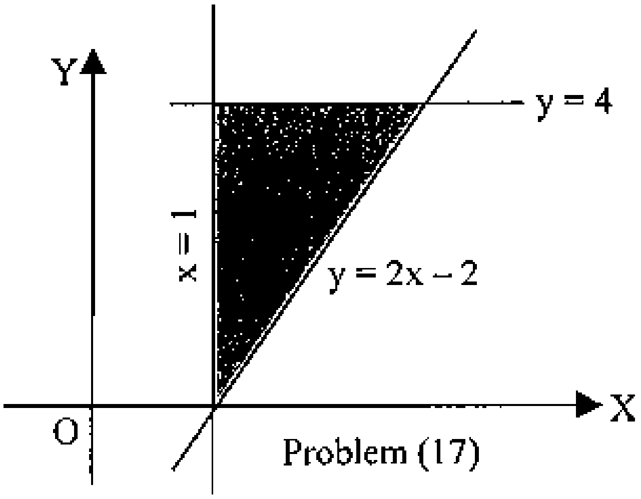


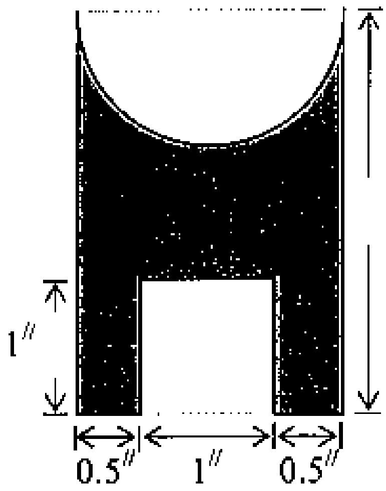
Problem (16)



Problem (17)

Problem (18): Determine centriodal moment of inertias ($I_{\bar{x}}$ & $I_{\bar{y}}$).

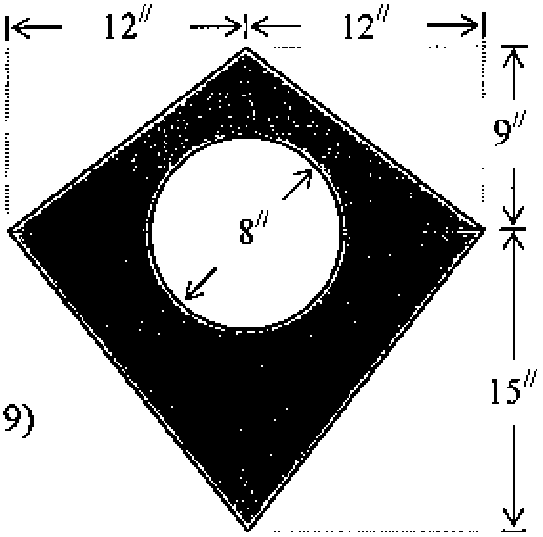
Solution (18): $I_{\bar{x}} = 1.351028905$ inch^4; $I_{\bar{y}} = 1.523967585$ inch^4.



Problem (18)

Problem (19): Determine centriodal moment of inertias ($I_{\bar{x}}$ & $I_{\bar{y}}$).

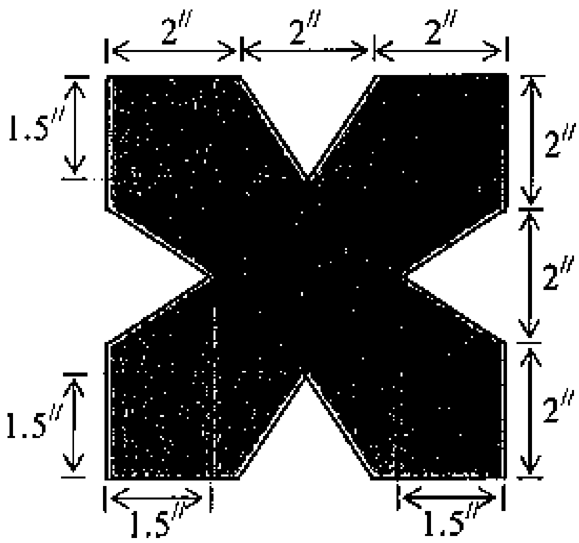
Solution (19): $I_{\bar{x}} = 6611.36454$ inch^4; $I_{\bar{y}} = 6710.93807$ inch^4.



Problem (19)

Problem (20): Determine centriodal moment of inertias ($I_{\bar{x}}$ & $I_{\bar{y}}$).

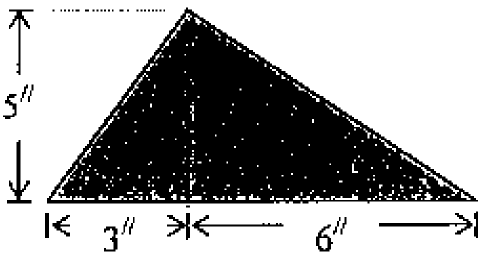
Solution (20): $I_{\bar{x}} = I_{\bar{y}} = 128.375$ inch^4.



Problem (20)

Problem (21): Determine centriodal moment of inertias ($I_{\bar{x}}$ & $I_{\bar{y}}$).

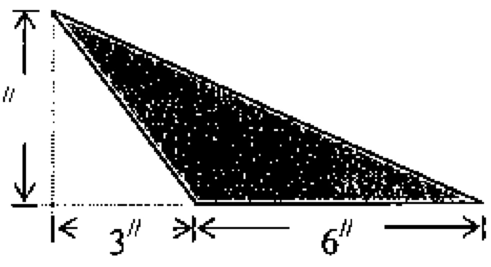
Solution (21): $I_{\bar{x}}$ = 72 inch^4; $I_{\bar{y}}$ = 456.75 inch^4.



Problem (21)

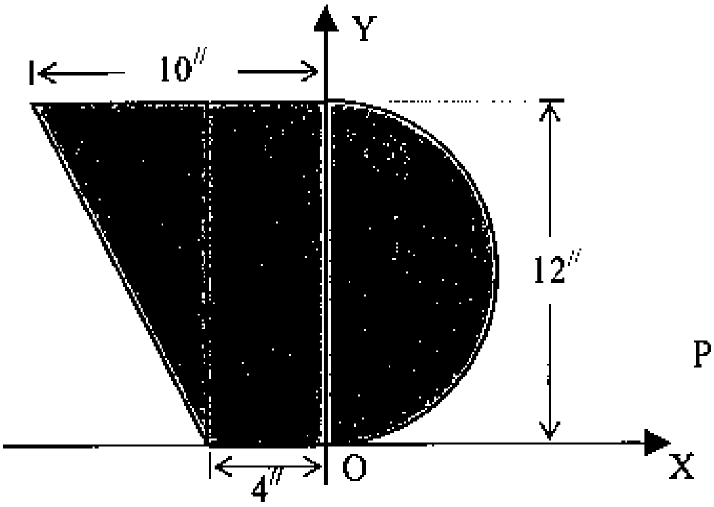
Problem (22): Determine centriodal moment of inertias ($I_{\bar{x}}$ & $I_{\bar{y}}$).

Solution (22): $I_{\bar{x}}$ =90 inch^4; $I_{\bar{y}}$ = 1272.50 inch^4.



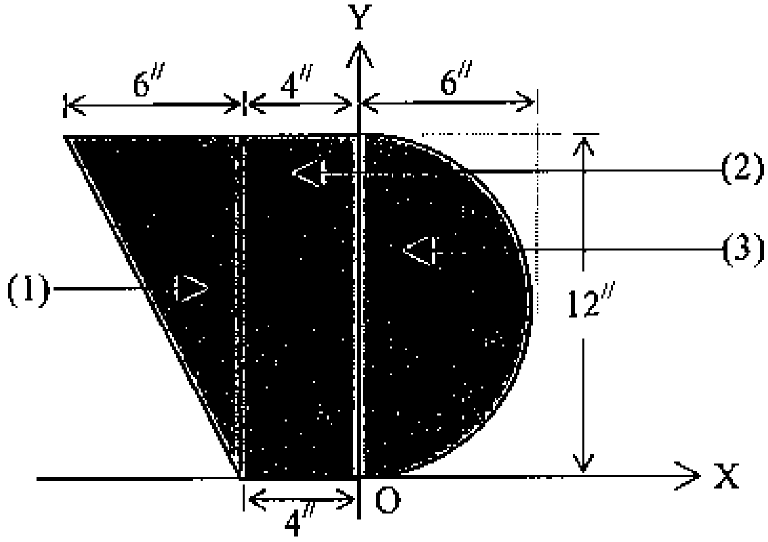
Problem (22)

Problem (23): Determine moments of inertias with respect to X & Y axes and product of inertia with respect to theses set of axes. Also, determine minimum radius of gyration with respect to XY axis about point O. (Linear unit is in inch).



Problem (23)

Solution (23):



$A_1 = 36 \text{ inch}^2;\ x_1 = -6 \text{ inch};\ y_1 = +8 \text{ inch}$

$A_2 = 48 \text{ inch}^2;\ x_2 = -2 \text{ inch};\ y_2 = +6 \text{ inch}$

$A_3 = \dfrac{\pi(6)^2}{2} \text{ inch}^2 = 18\pi \text{ inch}^2;\ x_3 = +\dfrac{4(6)}{3\pi} \text{ inch} = +\dfrac{8}{\pi} \text{ inch};\ y_3 = +6 \text{ inch}$

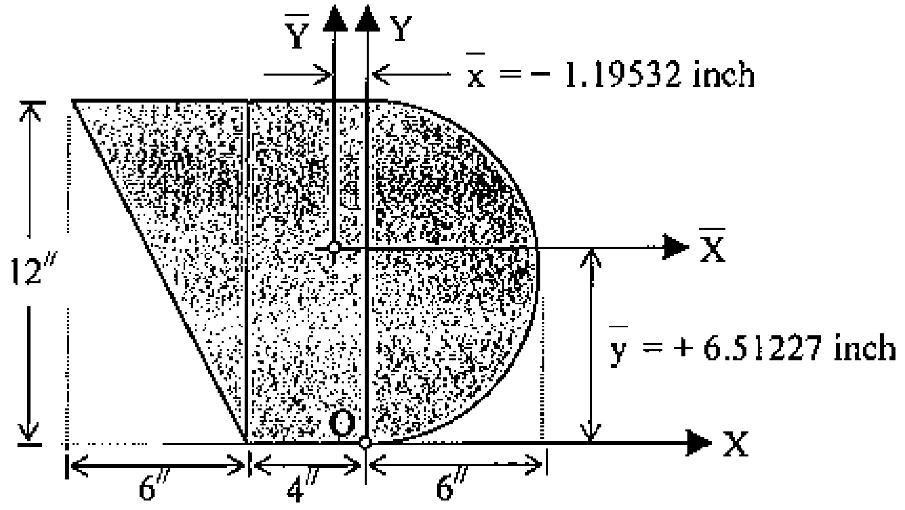
$A = A_1 + A_2 + A_3 = 140.54866 \text{ inch}^2$

$$\Rightarrow \bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = \frac{(36)\times(-6) + (48)\times(-2) + 18\pi\times\left(\dfrac{8}{\pi}\right)}{(36) + (48) + 18\pi}$$

$\Rightarrow \bar{x} = -1.195315 \text{ inch}$

$$\Rightarrow \bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = \frac{(36)\times(8) + (48)\times(6) + 18\pi\times(6)}{(36) + (48) + 18\pi}$$

$\Rightarrow \bar{y} = -6.51227 \text{ inch}$



$$I_{\bar{x}} = I_{x1} + I_{x2} + I_{x3}$$

$$= \left[\overline{I_{x1}} + A_1(y_1 - \bar{y})^2\right] + \left[\overline{I_{x2}} + A_2(y_2 - \bar{y})^2\right] + \left[\overline{I_{x3}} + A_3(y_3 - \bar{y})^2\right]$$

$$I_{\bar{x}} = \begin{array}{l} \left[\dfrac{6(12)^3}{36} + 36\times(8-6.51227)^2\right] + \left[\dfrac{4(12)^3}{12} + 48\times(6.51227-6)^2\right] \\ + \left[\dfrac{\pi(6)^4}{8} + 18\pi\times(6.51227-6)^2\right] = 1480.05399\,\text{inch}^4 \end{array}$$

$$I_{\bar{y}} = I_{y1} + I_{y2} + I_{y3}$$

$$= \left[\overline{I_{y1}} + A_1(x_1 - \bar{x})^2\right] + \left[\overline{I_{y2}} + A_2(x_2 - \bar{x})^2\right] + \left[\overline{I_{y3}} + A_3(x_3 - \bar{x})^2\right]$$

$$I_{\bar{y}} =$$

$$\left[\frac{12(6)^3}{36} + 36\times\{-1.195315-(-6)\}^2\right] + \left[\frac{12(4)^3}{12} + 48\times\{-1.195315-(-2)\}^2\right]$$

$$+\left[0.10975696\times(6)^4 + 18\pi\times\left\{-1.195315-(+\frac{8}{\pi})\right\}^2\right] = 1932.13385\,\text{inch}^4$$

$$I_x = \overline{I_x} + A\,(\Delta y)^2 = 1480.05399 + 140.54866\times(6.51227)^2 = 7440.6749\ \text{inch}^4$$

$$I_y = \overline{I_y} + A\,(\Delta x)^2 = 1932.13385 + 140.54866\times(1.19532)^2 = 2132.9450\ \text{inch}^4$$

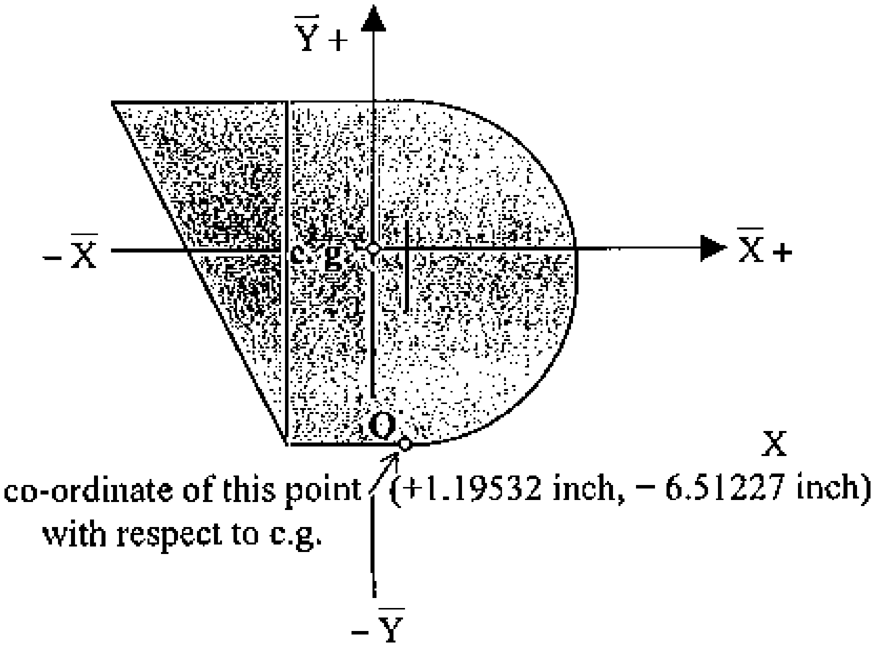
$$P_{\bar{x}\bar{y}} = P_{xy1} + P_{xy2} + P_{xy3}$$

$$P_{\overline{xy}} = \left[\overline{P_{xy1}} + A_1(x_1-\bar{x})(y_1-\bar{y})\right] + \left[\overline{P_{xy2}} + A_2(x_2-\bar{x})(y_2-\bar{y})\right] + \left[\overline{P_{xy3}} + A_3(x_3-\bar{x})(y_3-\bar{y})\right]$$

$$P_{\overline{xy}} = [0+36\times\{-6-(-1.195315)\}]\times\{8-6.51227\} + [0+48\times\{-2-(-1.195315)\}\times\{6-6.51227\}] + \left[0+18\pi\times\left\{\frac{8}{\pi}-1.195315\right\}\times\{6-6.51227\}\right] = -345.93499\text{inch}^4$$

$P_{xy} = \overline{P_{xy}} + A(\Delta x)(\Delta y) = -\ 345.93499 + 140.54866\times(+1.195315)\times(-6.51227)$

$= -\ 1439.99126\ \text{inch}^4$



$\overline{Y}+$
$-\overline{X}$
$\overline{X}+$
c.g.
O
X
co-ordinate of this point (+1.19532 inch, − 6.51227 inch) with respect to c.g.
$-\overline{Y}$

alternatively, $P_{\overline{xy}} = P_{xy1} + P_{xy2} + P_{xy3}$

$$P_{\overline{xy}} = \left[\overline{P_{xy1}} + A_1x_1y_1\right] + \left[\overline{P_{xy2}} + A_2x_2y_2\right] + \left[\overline{P_{xy3}} + A_3x_3y_3\right]$$

$$P_{\overline{xy}} = [0+36(-6)(+8)] + [0+48(-2)(+6)] + [0+18\pi(8/\pi)(6)] = -\ 1439.126\ \text{inch}^4$$

$$I_{min} = \left(\frac{I_x + I_y}{2}\right) - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

$$=\left(\frac{7440.6749+2132.9450}{2}\right)-\sqrt{\left(\frac{7440.6749-2131.9450}{2}\right)^2+(-1439.99126)^2}$$

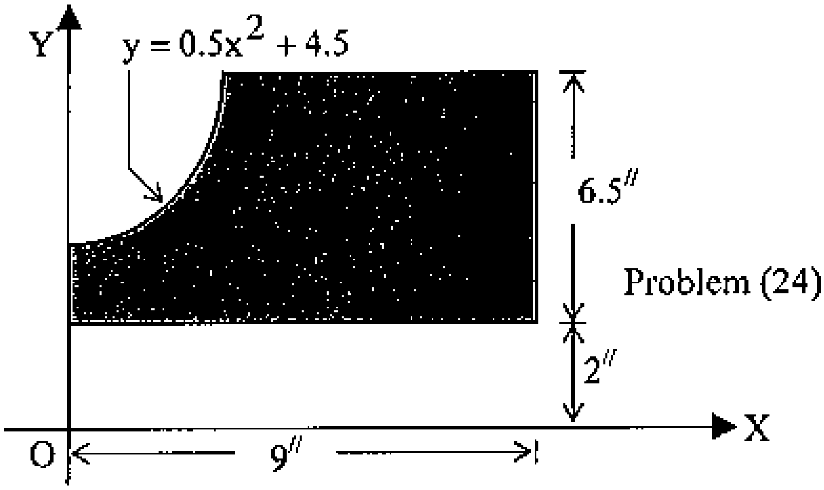
$= 1767.44267 \text{ inch}^4$

$$k_{min}(\text{about point "O"})=\sqrt{\frac{I_{min}}{A}}=\sqrt{\frac{1767.44267}{140.54866}}=3.546168\,\text{inch}$$

Answer (23): I_x = 7440.690049 inch^4; I_y = 2132.93801 inch^4; P_{xy} = −1439.9929 inch^4; k_{min} = 3.546158899 inch.

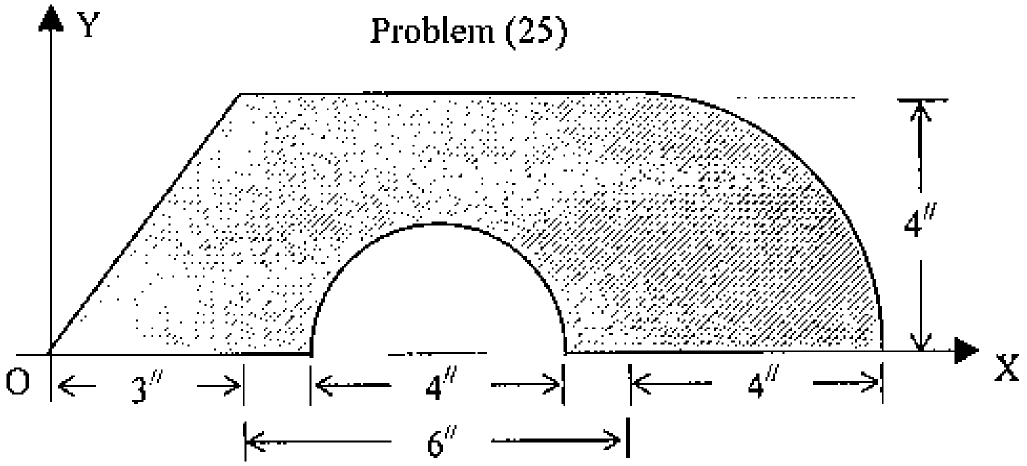
Problem (24): Determine moments of inertias with respect to X & Y axes and product of inertia with respect to theses set of axes. (Linear unit is in inch).

Answer (24): I_x = 494.96875 inch^4; I_y = 1567.432044 inch^4; P_{xy} = +1061.479167 inch^4;



Problem (25): Determine minimum radius of gyration (a) with respect to centroidal axis, (b) with respect to XY axis about point O (as shown).

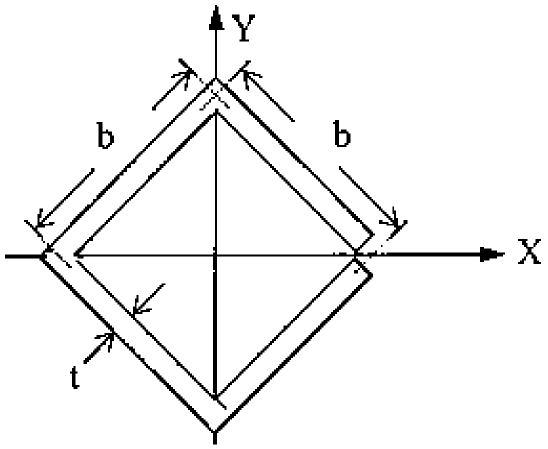
Answer (25): (a) 1.421574837 inch; (b) 4.287916343 inch.

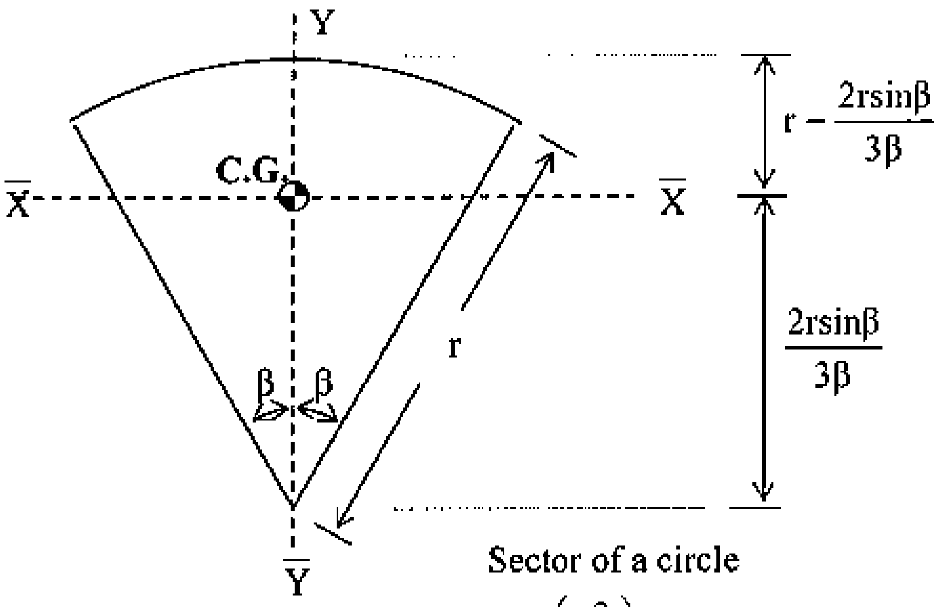


Problem (26): Compute moment of inertia with respect to X-axis.

Answer (26): $\frac{2b^3t}{3}+\frac{bt^3}{6}$

Problem (26)





Sector of a circle

Cross-sectional area, $A=\beta r^2=(2\beta)\left(\frac{r^2}{2}\right)$

$$\overline{I}_x=\frac{r^4}{4}\left(\beta+\sin\beta\cos\beta-\frac{16\sin^2\beta}{9\beta}\right)$$

$$\overline{I}_y=\frac{r^4}{4}(\beta-\sin\beta\cos\beta)$$

If $\beta=\frac{\pi}{2}$, sector of a circle is a semi-circle

$\overline{I}_x=0.10975696\ (r)^4$ and $\overline{I}_y=\frac{\pi r^4}{8}$

If $\beta=\pi$, sector of a circle is a full-circle

$\overline{I}_x=\frac{\pi r^4}{4}$ and $\overline{I}_y=\frac{\pi r^4}{4}$

Chapter Nine

Friction

Friction may be defined as the force distribution at a surface of contact between two bodies that prevents or impedes any possible sliding motion between the two bodies. This force distribution is tangent to the contact surface and has, for the body under consideration, a direction at every point in the contact surface that is in opposition to the possible or existing slipping motion of the body at that point.

Friction is both a boon and a bane to mankind. For example, without friction we should not able to walk; we could not get our cars in motion, nor could we stop them if they we were moving. Friction is due to the resistance offered to motion by minutely projecting particles at the contact surfaces.

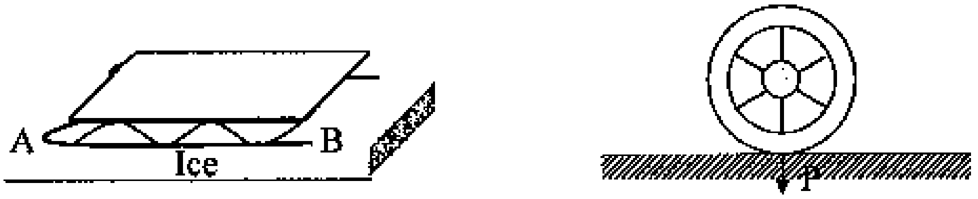
Frictional force always acts in a direction to oppose the motion that occurs or tends to occur between two surfaces.

Static Friction: The frictional resistance between two surfaces which have no relative motion is called static friction.

Kinetic Friction: The frictional resistance between two sliding surfaces which do move relative to each other is called kinetic friction (also called dynamic friction). It is two types:

(a) Sliding friction: It is the friction experienced by a body when it slides over other body.

(b) Rolling friction: It is the friction experienced by a body when it rolls over a surface.



(a) Sliding friction Figure 1 (b) Rolling friction

We could drag a cart over the ground by using sleds (and these happen to be very convenient over ice) but this means much hard work against friction. But a wheel instead of a sled at once transforms the problem because, instead of a broad contacting surface AB (figure 1(a)) such as the sled involves, we have at any instant only a single point of contact P in the case of the wheel (figure 1(b)). The sled produces sliding friction and the wheel rolling friction. It is always easier to move a body under rolling friction than a body under sliding friction.

Smooth & Rough Surfaces: A surface which offer frictional resistance is said to be rough, as opposed a frictionless surface which is said to be smooth.

It is actually impossible to have an accurately smooth surface. However smooth it seems to be, under a powerful microscope, the tiny unevennesses of the surface show themselves; and so even a very smooth ball moving over very smooth ice will ultimately come to a halt because of the frictional resistances that are created by these unevennesses.

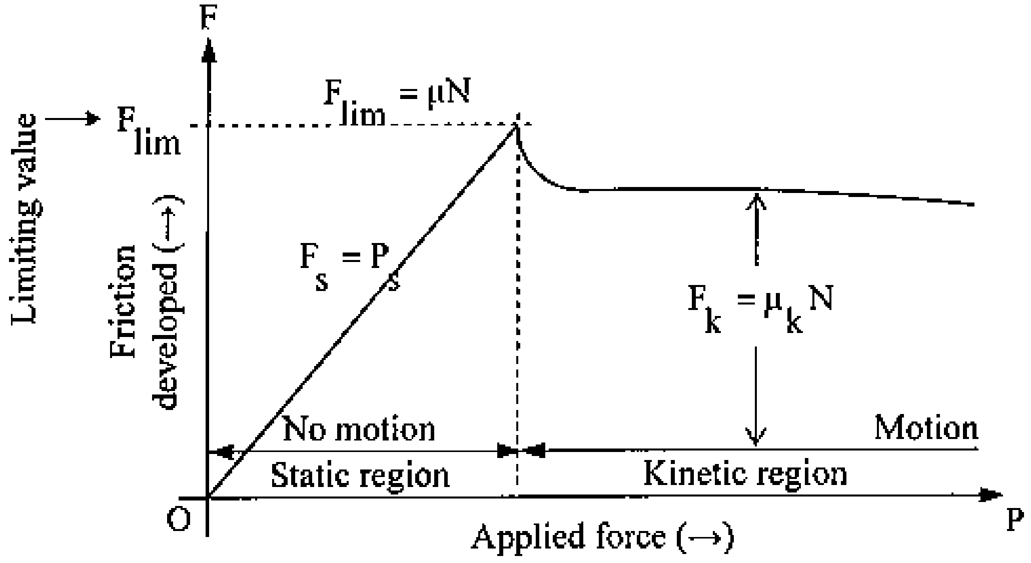


Figure 2(a)

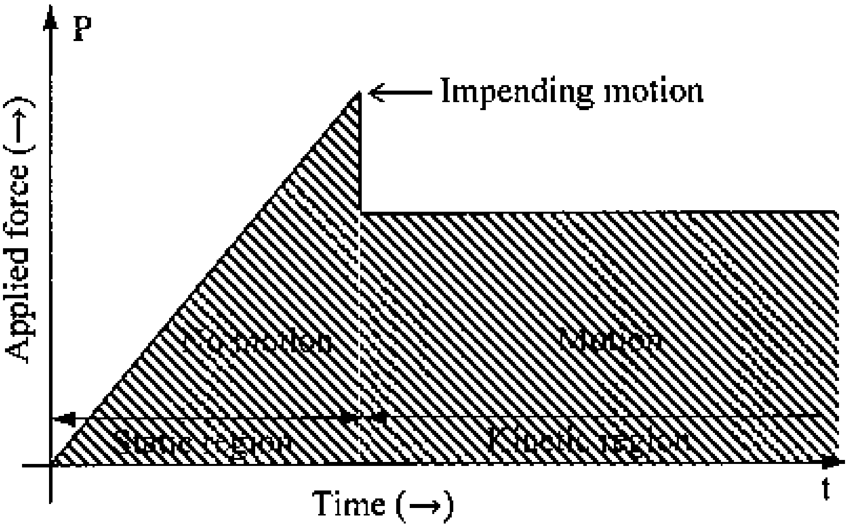


Figure 2(b)

Frictional force has the remarkable property of adjusting itself in magnitude to the force producing or tending to produce the motion so that the motion id prevented.

However, there is a limit beyond which the magnitude of this force can not increase. If the applied force is more than this maximum frictional force, there will be movement of one body over the other. This maximum value of frictional force, which comes into play, when the motion is impending, is known as **limiting friction** (see figure 2). It may be noted that when the applied force is less than the limiting friction, the body remains at rest and such frictional force is called static friction, which may have any value between zero and the limiting friction. If the value of the applied force exceeds the limiting friction, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as kinetic friction (or dynamic friction). Dynamic friction is found to be less than limiting friction.

[It can be shown experimentally that the force of limiting friction –

(a) Has nothing to do with the area of the contact surfaces in contact.
(b) Is different for different kind of surfaces.

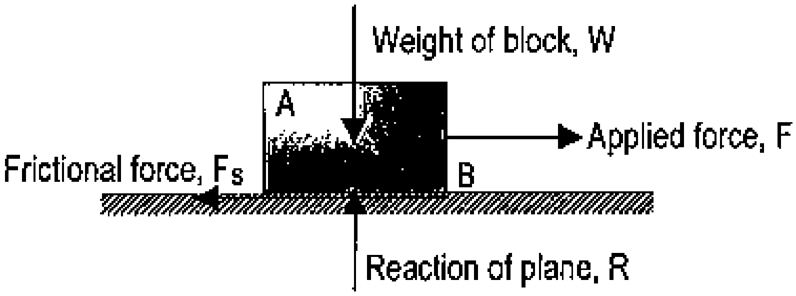


Figure 3: Limiting friction

(c) Is different for different weight W of the surface to be moved. This after all, is natural since the more weight there is pressing down on the surface B, the more opposition (i.e., friction) there must be to the pushing force F.
(d) But once the movement has started, and we pass from what we might call static friction to sliding friction, the amount of this diminishes below the critical level we spoke as limiting friction. In other words, at high speeds the "stopping" effect of brakes is much less than at lower speeds. For example, the force of friction between the brake drums and brake lining is a motor vehicle is more than twice as much at 20 miles per hour as at 60 miles per hour.]

Laws of Dry Friction: Coulomb (1781) & Morin (1831)

(a) The force of friction always acts in a direction opposite to that in which the body tends to move.
(b) Till the limiting value id reached, the magnitude of friction is exactly equal to the force which tends to move the body.
(c) The magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces.
(d) The force of friction depends upon the roughness/smoothness of the surfaces.
(e) The force of friction is independent of the area of contact between the two surfaces.

Types of Friction:

(a) Dry friction (Unlubricated friction)
(b) Wet friction (Viscous friction or lubricated friction)

The smallest practical coefficient of dry friction is about 0.02 (2 percent) for smooth, well-greased metal surfaces, while on an automobile tire on a dry concrete road has a friction coefficient of about 1.00 (100 percent).

Table 1: Coefficients of friction

Surfaces	Coefficient of static friction (μ_s)	Coefficient of kinetic friction (μ_k)
cast iron on cast iron	0.2	0.15
Leather on iron	0.5	0.3
Leather on wood	0.4	0.3
Steel runner on snow and ice	---	0.02
Steel on steel, clean & dry	0.8	---
Steel on steel, surface oxidized	0.3	---
Steel on steel, wet with oil	---	0.2
Steel on steel, lubricated with graphite	---	0.06
Tire on dry asphalt	0.7	
Wood on wood, dry	0.5	0.3
Wood on cast iron, dry	0.6	0.33

Experiment show that the coefficient of kinetic friction (μ_k) for a particular pair of surfaces is lower than the coefficient of static friction (μ_s).

Laws of Friction: Experiments carried on by scientists and engineers for more than 150 years have led to the following general conclusions, known as the laws of friction.

Dry surfaces	Lubricated friction
(1) the friction force is dependent on the kinds of the materials and the degree of roughness of the two surfaces in contact.	(1) the friction force is not dependent on the kinds of materials nor (with limits) on the degree of roughness of the contacting surfaces.
(2) the maximum friction force (limiting friction) which can be developed is dependent upon the normal pressure between the contacting surfaces, and is proportional to it.	(2) the friction force is not dependent upon the normal pressure between the contacting surfaces.

(3) the friction force is not dependent upon the areas of the surfaces in contact.	(3) the friction force is dependent upon the areas of the surfaces in contact.
(4) the friction force is dependent on the velocity (to a limiting extent), and decreases as the velocity increases.	(4) the friction force is dependent on the velocity and increases as the velocity increases.
(5) the friction force is not dependent on temperature in any important degree.	(5) the friction force is dependent on temperature to a considrable degree.

Determination of Co-efficient of Static Friction (μ_s): The simples way determine experimentally the value of a co-efficient of static friction (μ_s) is by means of an inclined plane (see figure 4). The plane is lined with one substance and the block on it with the other substance. The angle of inclination of the plane is slowly increased until the block starts to slide down. From the angle θ at which this occurs, the value of μ_s can be determined. On the block there are three forces acting as shown. The weight passes through the center of gravity, and hence the reaction from the plane also passes through that point. It is resolved N and F and from the equilibrium conditions, before sliding, we see that

$$N = W\cos\theta$$
$$N = W\cos\theta$$
$$N = F\tan\theta$$

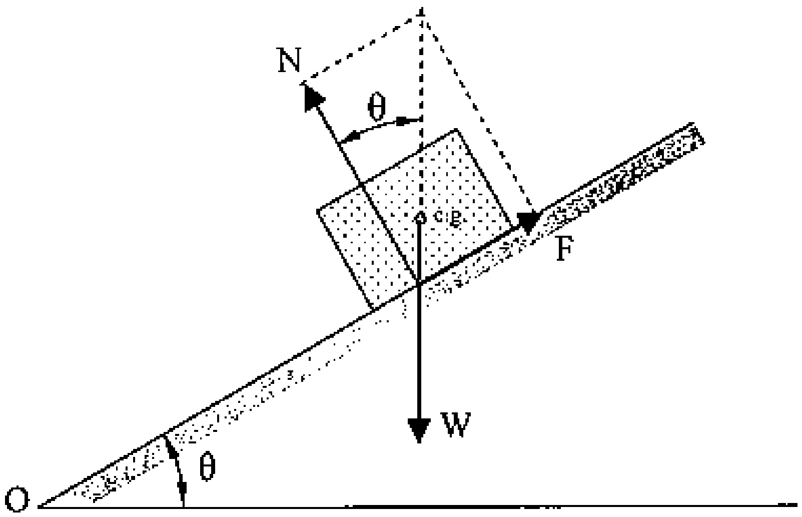


Figure 4: Equilibrium of block on a rough inclined plane

The angle between the total contact force W and its normal component N, when sliding just starts, is called the friction angle, which in this case equals the angle of inclination of the plane. Thus

$$F = \mu_s N$$

$$\mu_s = \frac{F}{N} = \tan\theta$$

The friction coefficient equals the tangent of the friction angle, which again is equal to the tangent of the inclination of the plane.

Question: Co-efficient of static friction (μ_s) between a rubber tire and a concrete road is 1.00. What does it mean?

Answer:

$$\mu_s = \tan\theta$$

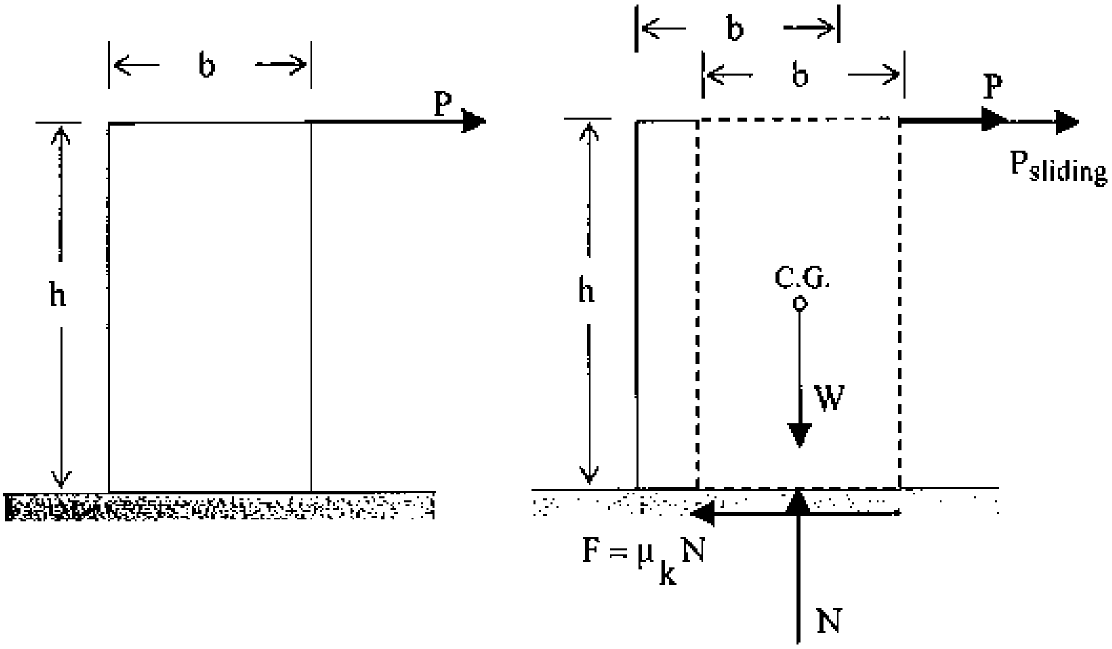
$$\theta = \tan^{-1}(\mu_s) = \tan^{-1}(1.00) = 45^0$$

Co-efficient of static friction (μ_s) = 1.00 between a rubber tire and a concrete road means that a car with completely locked brakes can stand on a 45-degree incline and that when the incline becomes slightly steeper, the car will start sliding down with locked wheels.

Impending Motion: Two types of impending motion

(a) Slipping or sliding

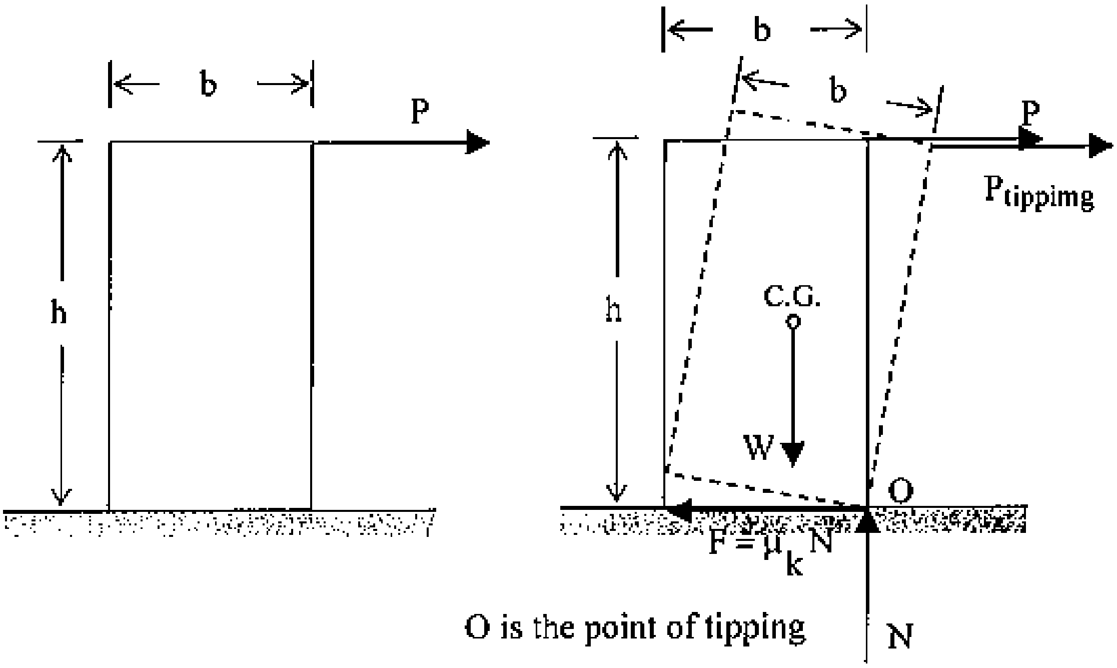
(b) Tipping or overturning or rotation



(a) Before sliding

(b) After sliding (position of body is shown by dotted line)

Figure 5: Sliding or slipping



(a) Before tipping

(b) After tipping (position of body is shown by dotted line)

Figure 6: Tipping or overturning

Impending tipping	Impending slipping
N and F act the tipping edge	Location of N is unknown
$F_{max} = \mu_s N$; $F = F_{max}$	$F_{max} = \mu_s N$; $F \le F_{max}$

Find P in both cases; if

$P_{sliding} > P_{tipping}$; Tipping governs (Tipping occurs before sliding)

$P_{tipping} > P_{sliding}$; Sliding governs (Sliding occurs before tipping)

Some practical application of friction:

(a) Wedges
(b) Jamming or self-locking
(c) Screws
(d) The ladder against the wall
(e) Puling a block up the inclined plane
(f) The buggy wheel
(g) The differential hoist
(h) The pipe wrench and the automobile brake
(i) Friction in thrust bearings and clutches
(j) Belt & cable friction

Wedges: Wedges are small pieces of materials with two of its opposite surfaces not parallel. They are used to lift heavy blocks, precast beams etc, or adjusting levels of heavy machinery by the application of relatively small forces.

Belt Friction: Belts are often used for the transmission of a torque from one shaft to another and have some advantages over rearing; transmission is carried out by having cylindrical pulleys fixed to each shaft and a continuous belt passing around the rims of pulleys.

Friction is necessary between the belt and the pulleys in order that a torque can be transmitted, and the maximum torque that can be transmitted is related to the coefficient of friction between them.

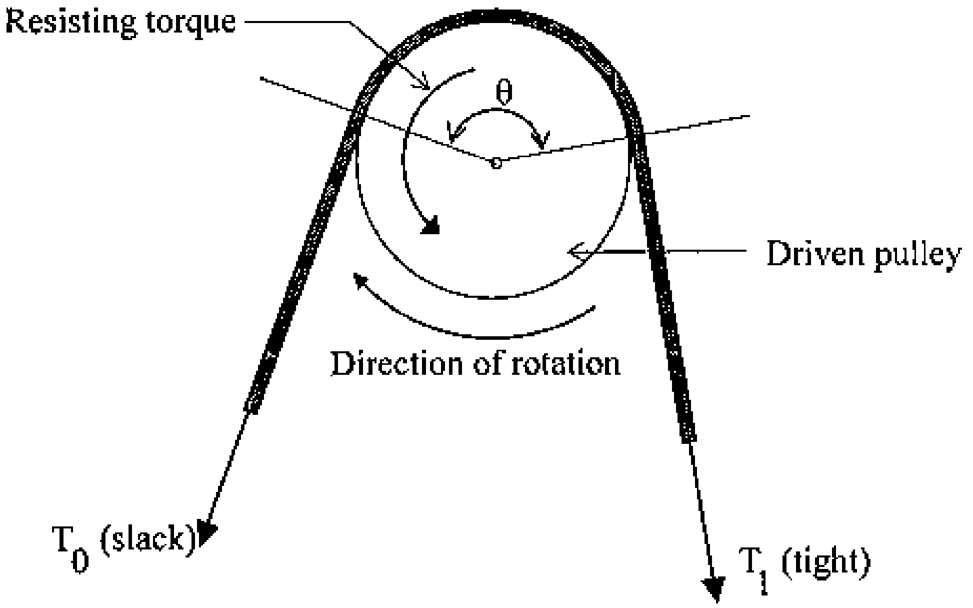


Figure 7: Belt friction

The equations of equilibrium, summing along the tangent and perpendicular to it

$$\sum F_{\text{(in tangential direction)}} = 0$$

$$(T + dT)\cos\frac{d\theta}{2} - T\cos\frac{d\theta}{2} - \mu dN = 0$$

$$\sum F_{\text{(in radial direction)}} = 0$$

$$dN - (T + dT)\sin\frac{d\theta}{2} - T\sin\frac{d\theta}{2} = 0$$

Since $\frac{d\theta}{2}$ is a small angle, $\cos\frac{d\theta}{2} \approx 1$ and $\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$

$$(T + dT) - T - \mu dN = 0$$

$$dN - T\frac{d\theta}{2} + dT\frac{d\theta}{2} - T\frac{d\theta}{2} = 0$$

$$dN + dT\frac{d\theta}{2} - Td\theta = 0$$

The order of magnitude of the product of two differentials is so much smaller than the other terms that the term $dT\frac{d\theta}{2}$ will be neglected.

$$dT = \mu dN$$

$$dN = Td\theta$$

$$dT = \mu dN = \mu Td\theta$$

$$\frac{dT}{T} = \mu d\theta$$

$$\int_{T_0}^{T_1} \frac{dT}{T} = \int_0^\theta \mu d\theta$$

$$\ln\frac{T_1}{T_0} = \mu\theta$$

$$T_1 = T_0 e^{\mu\theta}$$

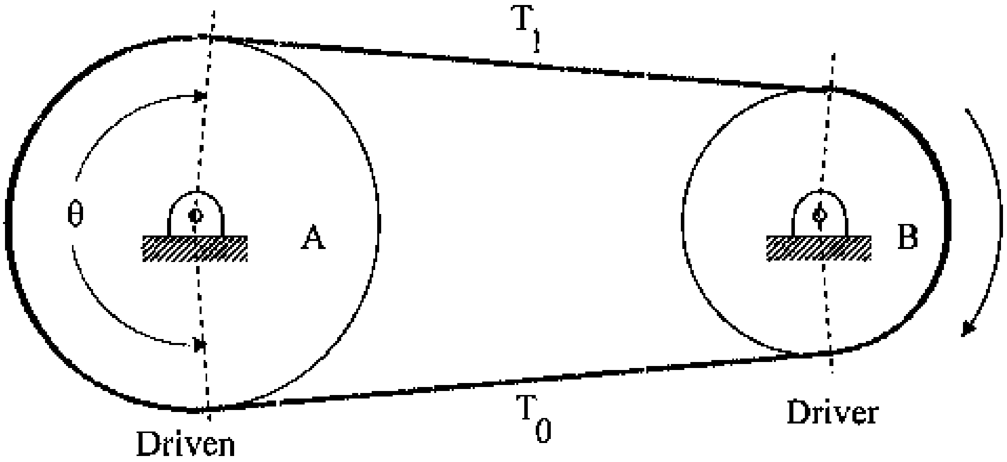
T_1 = tension on the tight side

T_0 = tension on the slack side

θ = angle of wrap of the belt on the drum (in radians)

μ = coefficient of friction between the belt and the drum

e = base of natural logarithms



Belts are used extensively in transmitting power from one shaft to another. In the simplest case, the belt passes around two pulleys, as shown in figure. Pulley B is the driving pulley; hence the upper belt has more tension in it than the bottom belt ($T_1 > T_0$). If friction did not exist between the belt and the pulleys, the driving pulley could not drive the belt, nor could the driven pulley be turned by the belt.

The same type of analysis may be applied for a fixed drum, if we assume that (i) the belt is flexible, (ii) it has negligible weight, (iii) low to moderate angular velocities prevail, and (iv) slip is impending.

$T_1 = T_0 e^{\mu\theta}$; here $T_1 > T_0$ [Note: θ must be in radians]

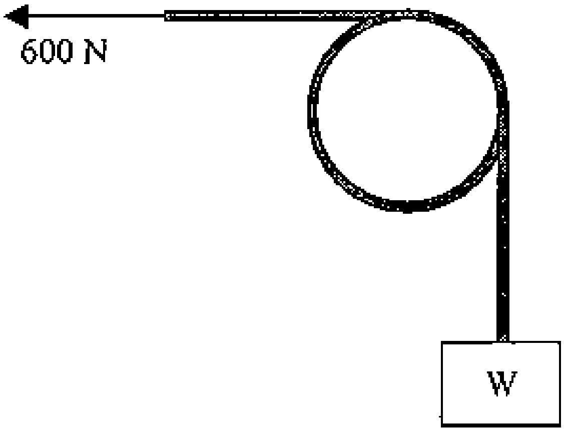
The total frictional force between the belt and the cylinder is

$T_1 - T_0 = T_1\left(1 - \frac{T_0}{T_1}\right) = T_1\left(1 - e^{-\mu\theta}\right) = T_0\left(e^{\mu\theta} - 1\right)$; These are the required relations between T_1 and T_0 when the belt is about to slip. The onset of slipping is therefore governed not by the difference in the tension but either their ratio, or their difference in comparison with the tension in one side or the other side of the belt.

$T_1 = T_0 e^{\mu\theta}$; gives the tension in the rope at any angular distance θ from the first contact point where ten tension is T_0. The tension rises like an e (e = exponential) function with the angle, and this rise is extremely rapid. As an example, let $\mu = 0.5$; then the table below shows the ratio of the rope tensions on the two sides of the drum for various angles.

θ (radians)	Times around drum	$\frac{T_1}{T_0}$
$\frac{\pi}{2}$	$\frac{1}{4}$	2.193280051
π	$\frac{1}{2}$	4.810477381
2π	1	23.14069263
4π	2	535.4916555
4π	3	12391.64781

Example: A rope making $1\frac{1}{4}$ turns around a stationary horizontal drum is used to support a weight W. If the coefficient of friction is 0.3, what range of weight W can be supported by exerting a 600 N force at the other end of the rope?



No. of turns around the drum = $1\frac{1}{4}$ & coefficient of friction (μ) = 0.3

Angle of contact (angle of wrap) = $1\frac{1}{4}\times 360^0 = 450^0 = 2.5\pi$ radians

(a) Let the impending motion of the weight (W) be downward, then
Tension in tight side $T_1 = W$

Tension in slack side $T_0 = 600$ N

$T_1 = T_0 e^{\mu\theta}$

$W = 600e^{(0.3\times 2.5\pi)}$

W = 6330.434444 N

(b) Let the impending motion of the weight (W) be upward, then
Tension in tight side $T_1 = 600$ N

Tension in slack side $T_0 = W$

$T_1 = T_0 e^{\mu\theta}$

$600 = We^{(0.3\times 2.5\pi)}$

$W = 600e^{-(0.3\times 2.5\pi)} = 56.86613491$ N

Thus, a 600 N force can support W = 56.86613491 N to W = 6330.434444 N weight on the other side of the drum.

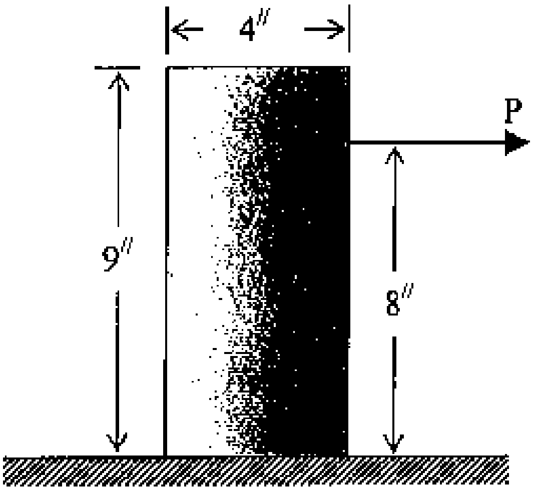
Problems

Problem (1): A 40-lb. ladder, 20 ft long, makes an angle of 60^0 with the ground. A vertical wall, against which the ladder rests, is smooth. The co-efficient of friction between the ladder and the horizontal ground is 0.50. How far up the ladder can a 160-lb man climb before motion impends?

Answer (1): 19.95527838 ft

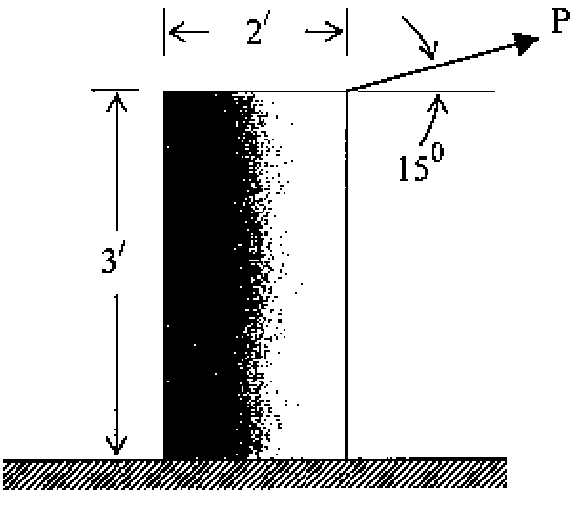
Problem (2): A 50-lb block, of negligible dimensions, is on a plane inclined 24^0 with the horizontal. What horizontal force P is necessary to start the motion up the plane if the coefficient of friction is 0.3?
Answer (2): P = 43 lb.

Problem (3): A 400-lb block rest on a horizontal surface. The coefficient of friction is 0.20. The force P is gradually increased. What maximum value can it have so that the block neither slide nor tip?
Answer (3): P = 80 lb.



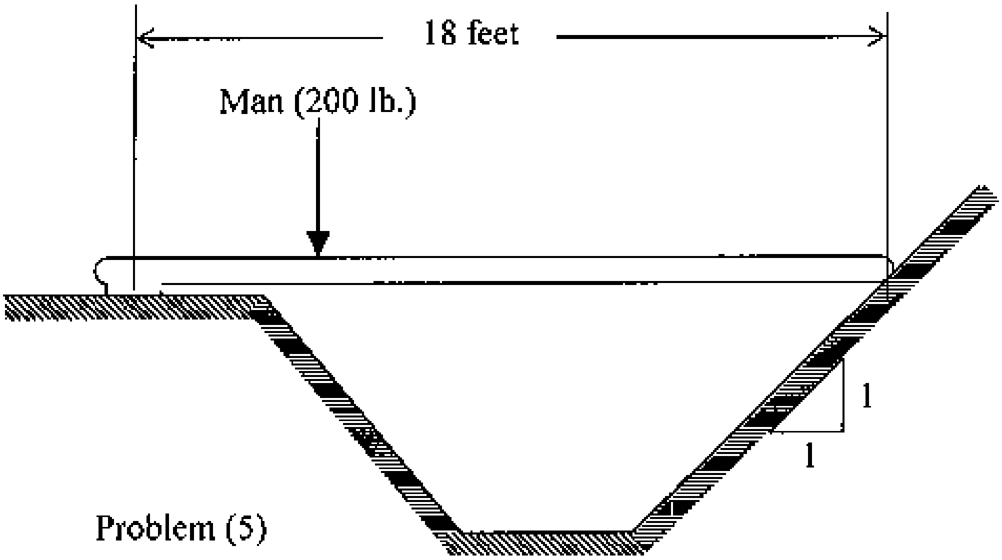
Problem (3)

Problem (3): Determine the value of P which will cause motion of the 200-lb homogeneous block to impend to the right. The coefficient of static friction between the block and the horizontal surface is 0.3.
Answer (3): P = 69.01841203 lb.



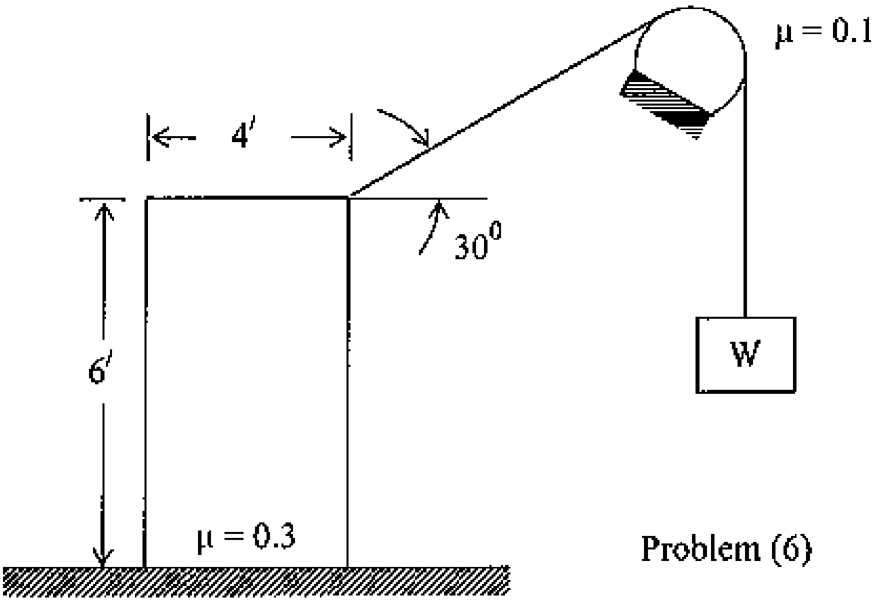
Problem (4)

Problem (5): How far can a 200-lb man walk out from the left on a weightless slab (see sketch) before slip occurs? The coefficient friction at all surfaces is 0.3.
Answer (5): 6.440366972 feet (from left)



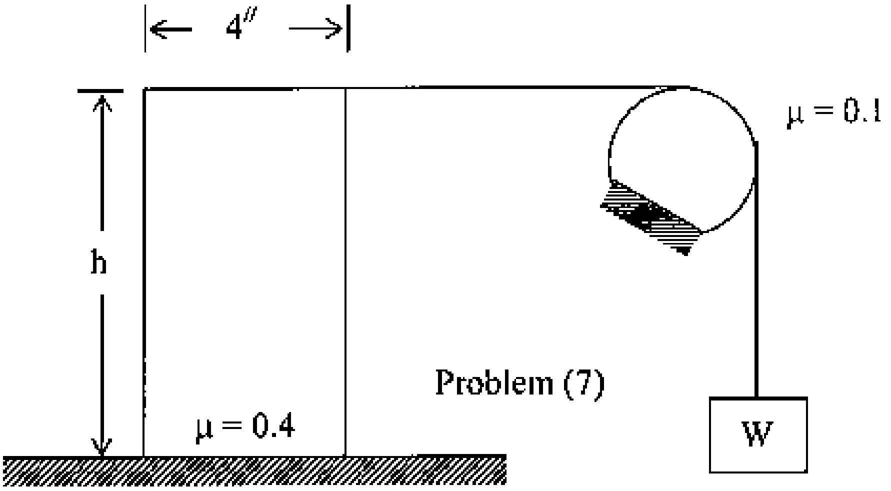
Problem (5)

Problem (6): A homogeneous block weighs 600 lb. Determine the minimum weight W for motion to impend. Will the block tip or slip?
Answer (6): W = 284.746102 lb.; the block will tip.



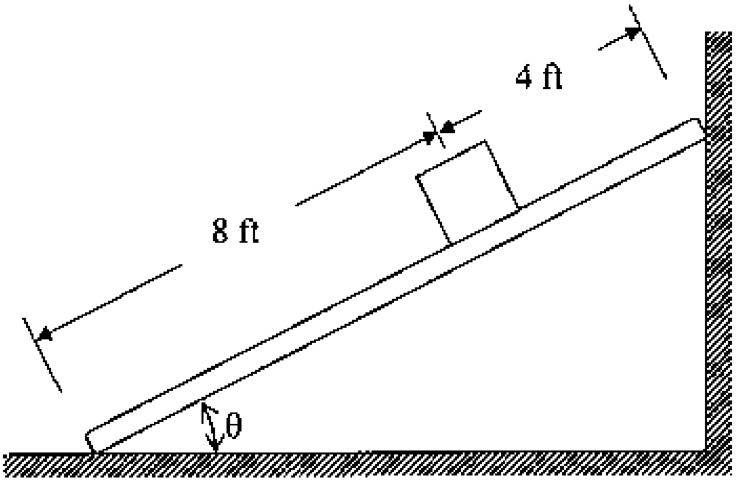
Problem (6)

Problem (7): Motion of a 20-lb block impends to the right on a horizontal plane for which the coefficient of friction is 0.40. A cord is horizontal from the block to a fixed drum over which it has an angle of wrap equal to 90^0. If the coefficient of friction between the cord and the drum is 0.10, determine (a) the weight W required to cause a sliding motion, and (b) the maximum value of h so that tipping will not occur.
Answer (7): (a) W = 9.3607103 lb. & (b) $h \leq 5$ inch.



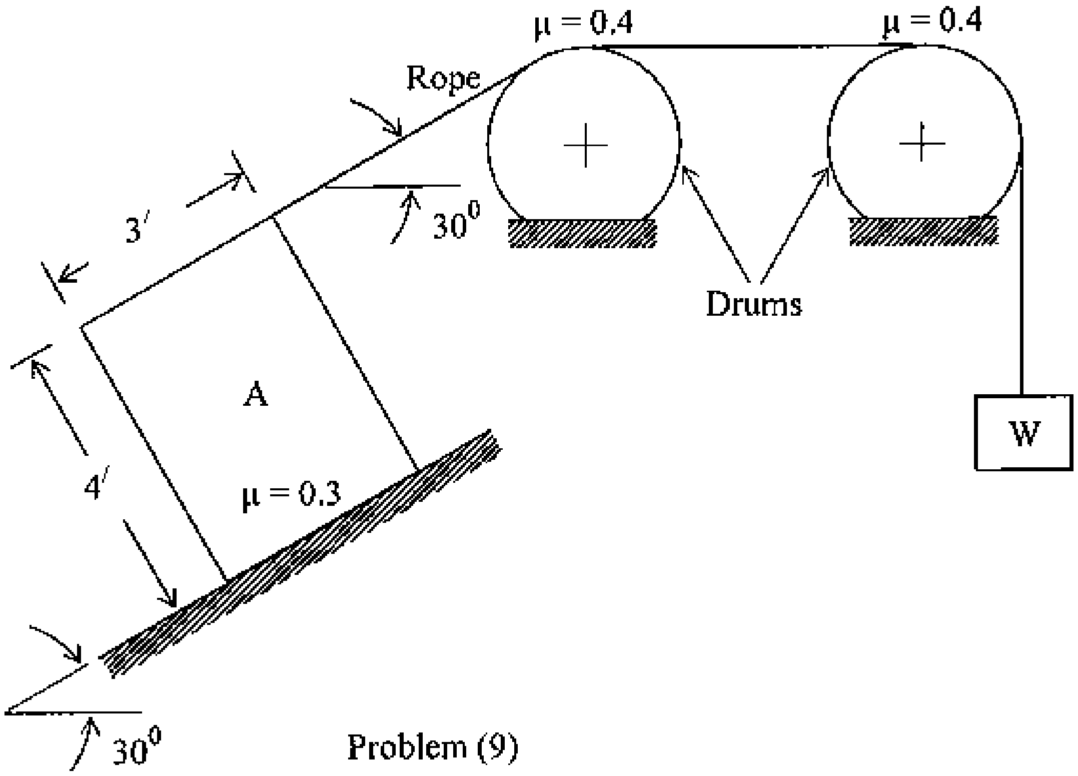
Problem (7)

Problem (8): a uniform plank weighs 40 lb and supports a 50-lb box. The coefficient of friction between the box and the plank is 0.50. What minimum coefficient of friction is required between the plank, the floor and the wall to prevent its slipping when motion impends between the box and the plank.
Answer (8): μ = 0.81749



Problem (8)

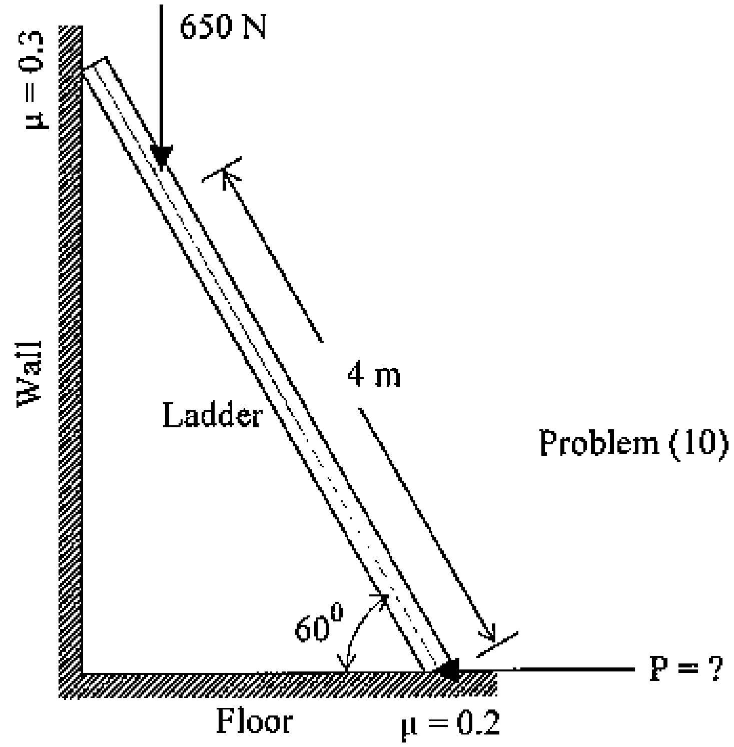
Problem (9): The body A weighs 250 lb. The coefficients of static friction are 0.3 between the body A and the inclined plane, and 0.4 between the rope and the drums. What value of W will cause motion of the body A to impend up the plane?
Answer (9) W = 622.4939746 lb.



Problem (9)

Problem (10): A ladder of length 5 meter and weighing 250 N is placed against a vertical wall. The coefficient of static friction between the wall and the ladder is 0.3 and that between the floor and the ladder is 0.20. The ladder also supports a man weighing 650 N. Calculate the minimum horizontal force P to be applied at the bottom of the ladder to prevent slipping of the ladder.

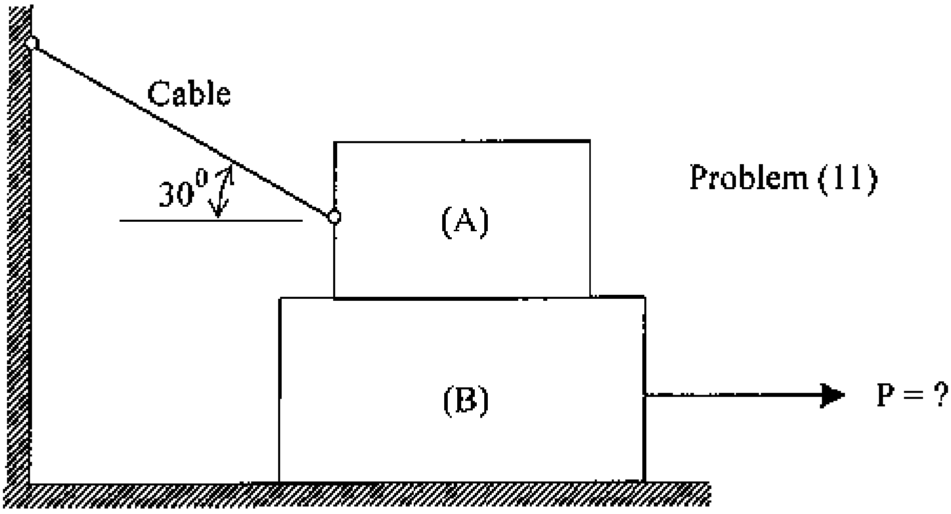
Answer (10): P = 156.4580324 lb. (←)



Problem (10)

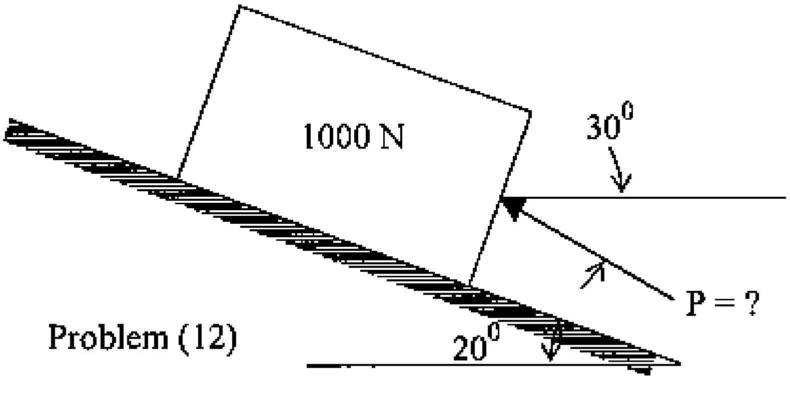
Problem (11): Block A weighing 300 N rests over the block B which weighs 900 N. Coefficient6 of friction for all contact surfaces is 0.3. What will be the value of the force P that will cause the block B to have impending motion toward right?

Answer (11): P = 423.4258613 lb. (→)



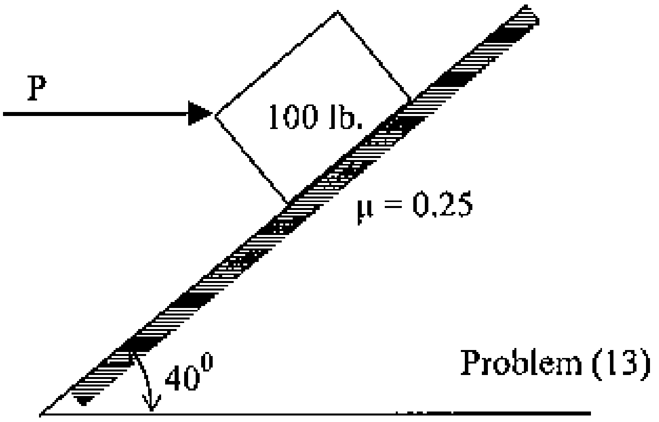
Problem (12): Determine the magnitude and direction of actual friction force acting on the 1000 N block if (i) P = 200 N and (ii) P = 600 N. The force P makes an angle of 30^0 with the horizontal. The values of the coefficients of static friction and kinetic friction are 0.25 and 0.20 respectively.

Answer (12): (i) 145.0585927 lb. (upward along the plane) & (ii) 208.8759386 lb. (downward along the plane)

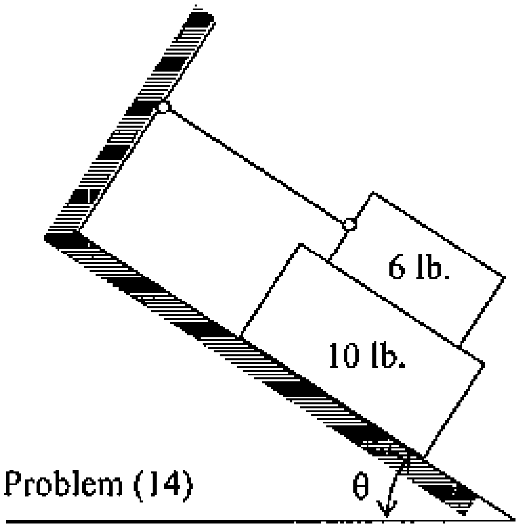


Problem (13): Determine the range of values which the horizontal force P may have without disturbing the equilibrium of the 100-lb block resting on the plane inclined at an angle of 40^0 with the horizontal. The coefficient of friction between the block and the plane is $\mu = 0.25$.

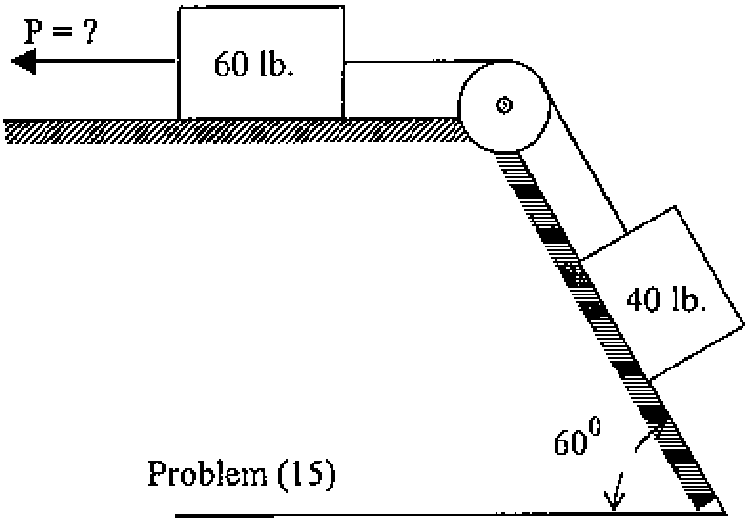
Answer (13): P = 48.69497849 lb. to 137.8214438 lb.



Problem (14): Determine what angle θ would be necessary for motion of the 10-lb block to impend down the plane. The upper block weighs 6 lb. and the coefficient of friction for all surfaces is 0.30. The rope is parallel to the plane.
Answer (14): θ = 33.42481118^0



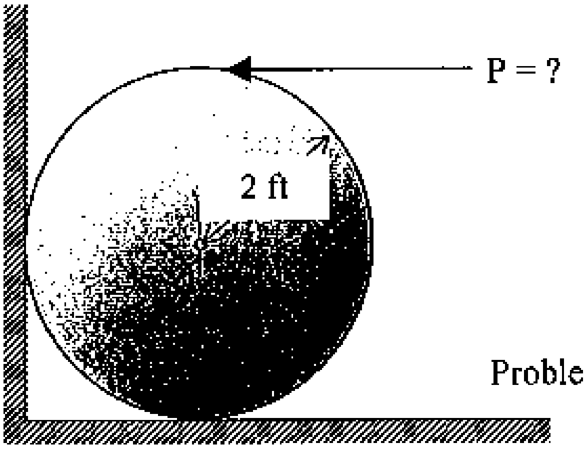
Problem (15): Determine the value of the horizontal force P is necessary to cause motion of the 60-lb block to impend to the left. The angle of friction is 18^0 for all surfaces. The rope passes over a smooth pulley. (Note: the tangent of the angle of friction has been defined as the coefficient of friction)
Answer (15): P = 60.63459185 (←)



Problem (15)

Problem (16): What magnitude of the horizontal force P will be necessary to cause the 120-lb. cylinder to move? The coefficient of friction for all surfaces is 0.40.

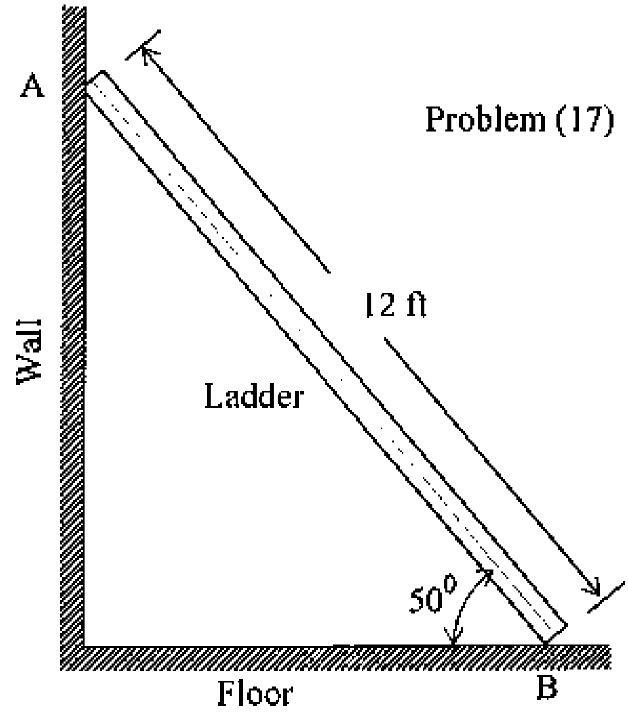
Answer (16): P = 73.04347826 lb.



Problem (16)

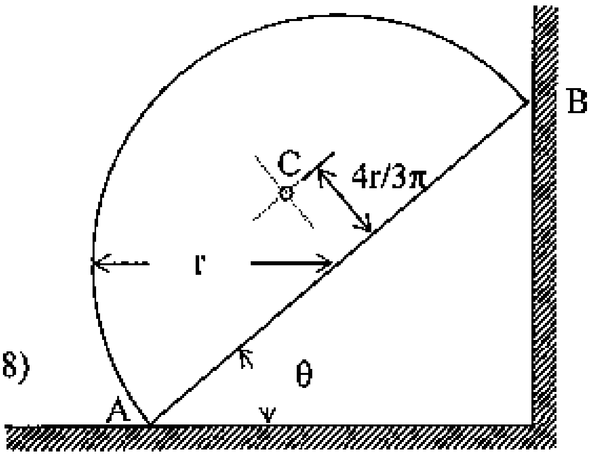
Problem (17): A homogeneous ladder AB (length 12 ft) weighs 22 lb. It rests on a horizontal floor and against the vertical wall. What must be the coefficient of friction μ for equilibrium?

Answer (17): μ = 0.363970234



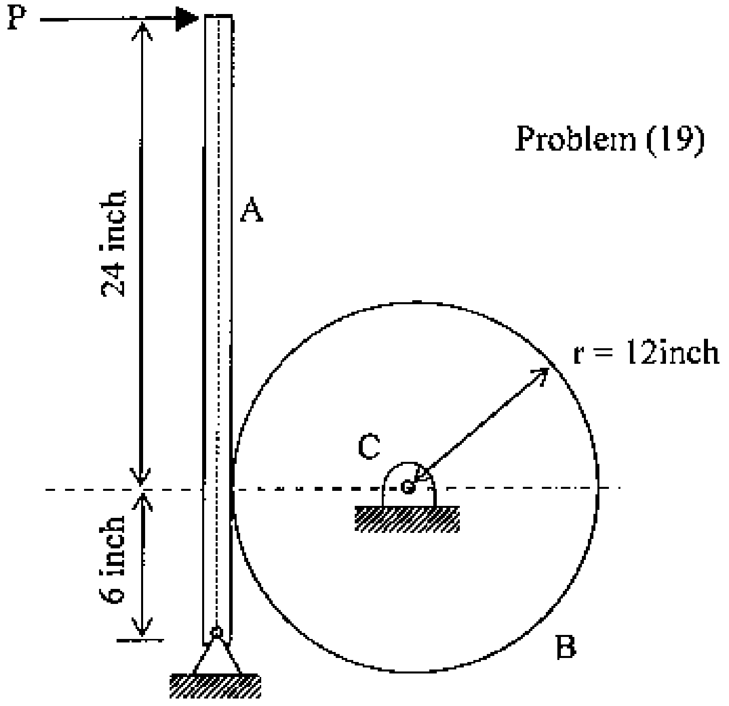
Problem (18): A half cylinder is on the verge of sliding. If the coefficient of friction for all surfaces is 0.3, determine the angle θ. The centroid C is located on the mid-radius at a distance 4r/3π from the diameter AB.

Answer (18): θ = 40.5761424°



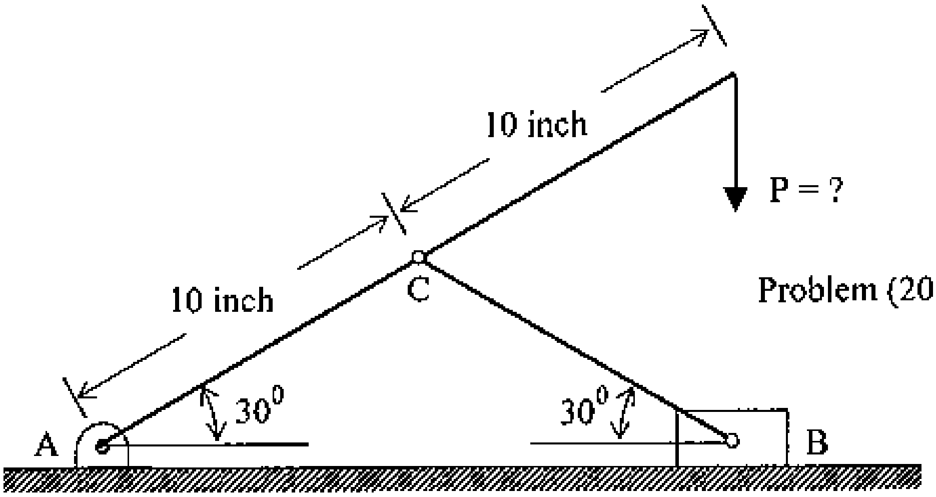
Problem (19): A vertical brake A is used to hold a circular drum B from rotating under the action of an 88-lb weight. (a) What is the magnitude of the horizontal force P if the coefficient of friction between A and B is 0.6? (b) What is the bearing reaction at C on the drum?

Answer (19): (a) P = 29.33333333 lb. (→); (b) R_{CX} = 146.6666667 lb. (←) & R_{CY} = 176 lb. (↑)



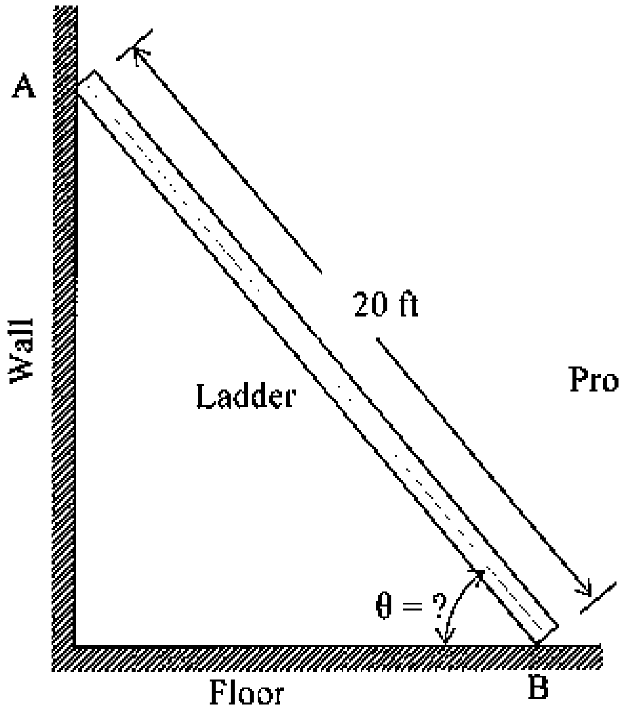
Problem (20): Determine the value of P for the motion of block B to impend if the coefficient of sliding friction between block B and the horizontal plane is 0.20. Block B weighs 18 lb and the force P is vertical. Neglect weight of the members.

Answer (20): P = 2.349791523 lb. (↓)



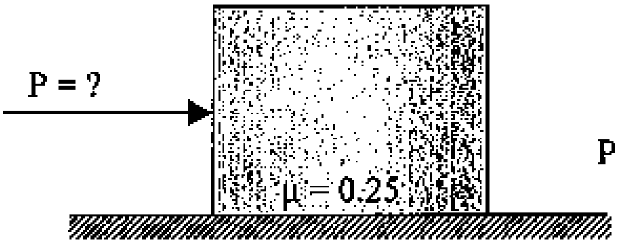
Problem (21): A 40-lb. ladder, 20 ft long, makes an angle of θ with the ground. A vertical wall, against which the ladder rests, is smooth. The co-efficient of friction between the ladder and the horizontal ground is 0.5. State the least value of the angle θ that would be required to stay in equilibrium.

Answer (21): $\theta = 45^0$



Problem (21)

Problem (22): Determine the horizontal force P necessary to cause motion of an 8-lb block to impend to the right. The coefficient of friction between the block and the horizontal plane is 0.25.

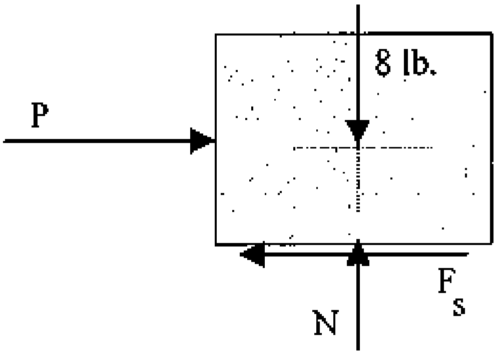


Problem (22)

Solution (2): N = reaction normal to the plane

F_s = frictional force (acts opposite to the direction of motion)

μ_s = co-efficient of static friction = 0.25



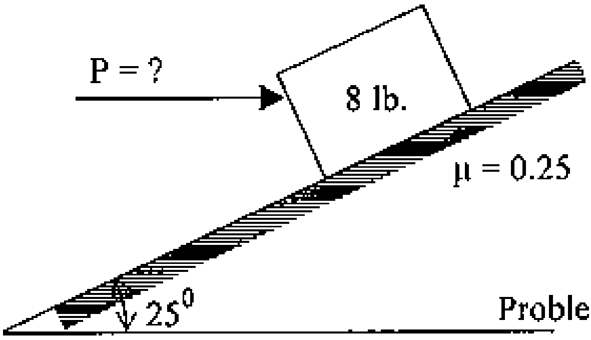
Free-body diagram of the block

Summing the forces in vertical direction, N = 8 lb. (↑)
From law of static friction, $F_s = \mu_s N$

$F_s = 0.25 \times 8 = 2$ lb. (←)
Summing the forces in horizontal direction direction, P = F_s = 2 lb. (→)

Answer (22): P = 2 lb. (→)

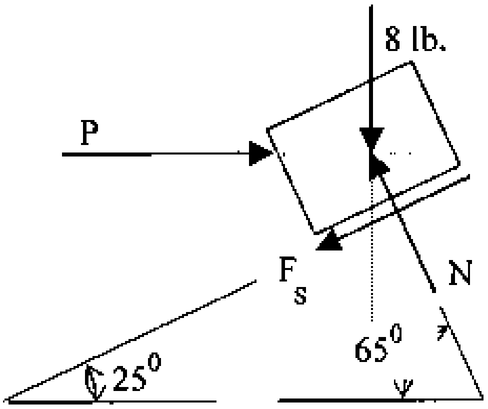
Problem (23): Determine the horizontal force P necessary to cause motion of an 8-lb block to impend to the right. The coefficient of friction between the block and the horizontal plane is 0.25.



Problem (23)

Solution (23): N = reaction normal to the plane
F_s = frictional force (acts opposite to the direction of motion)

μ_s = co-efficient of static friction = 0.25
From law of static friction, $F_s = \mu_s N = 0.25$ N



Free-body diagram of the block

Summing the forces in vertical direction (↑ +ve),
$\Rightarrow -8 + N\cos 25^0 - F_s \sin 25^0 = 0$

$\Rightarrow -8 + N\cos 25^0 - 0.25\,N\sin 25^0 = 0$
$\Rightarrow N = 9.991841392$ lb.

Summing the forces in horizontal direction (→ +ve),

$\Rightarrow + P - N \sin 25^0 - F_s \cos 25^0 = 0$

$\Rightarrow + P - N \sin 25^0 - 0.25\, N \cos 25^0 = 0$

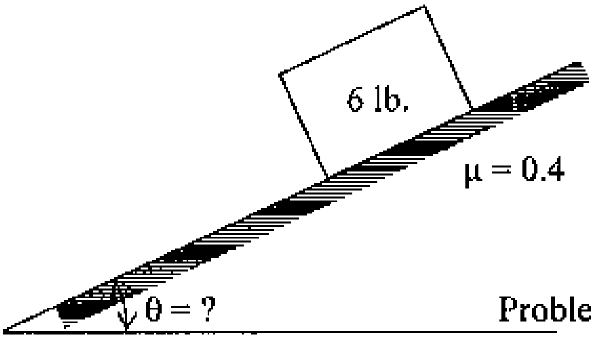
$\Rightarrow P = N \sin 25^0 + 0.25\, N \cos 25^0 = N (\sin 25^0 + 0.25 \cos 25^0)$

$\Rightarrow P = 9.991841392 \times (\sin 25^0 + 0.25 \cos 25^0) = + 6.486655556$ lb. (→)

Answer (23): $P = 6.486655556$ lb. (→)

Problem (24): A 6-lb block is at rest on a plane which can be inclined at an angle θ with the horizontal. The coefficient of friction between the block and the plane is assumed to be 0.4. What is the maximum angle θ to which the plane can be raised before motion impends?

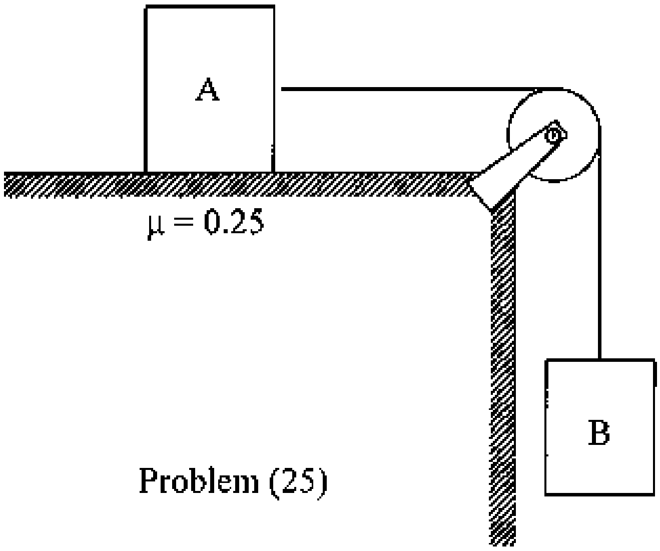
Answer (24): $\theta = 21.80140949^0$



Problem (24)

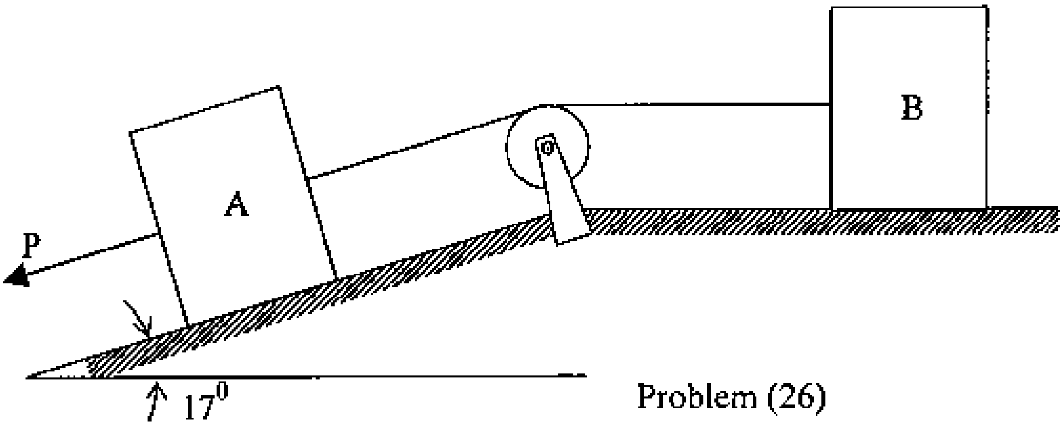
Problem (25): What is the maximum weight B can have before motion impends? Body A weighs 35 lb, and the coefficient of friction between A and The horizontal plane is 0.25.

Answer (25): 8.75 lb.



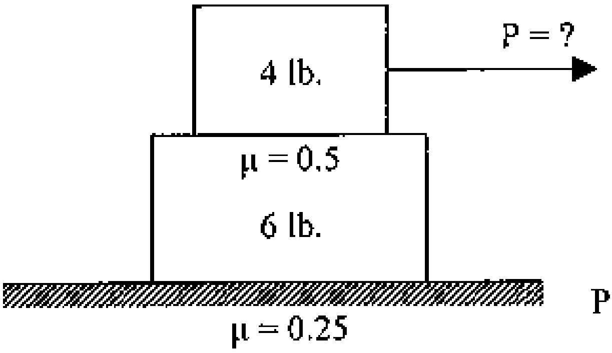
Problem (25)

Problem (26): What value of the force P parallel to the inclined plane is necessary to cause motion to impend down the plane? Body A weighs 10 lb. & body B weighs 8 lb and rests on a horizontal plane. The coefficient of friction for all surfaces is assumed to be 0.35. The cords are parallel to the planes.
Answer (26): P = 3.223349599 lb.



Problem (26)

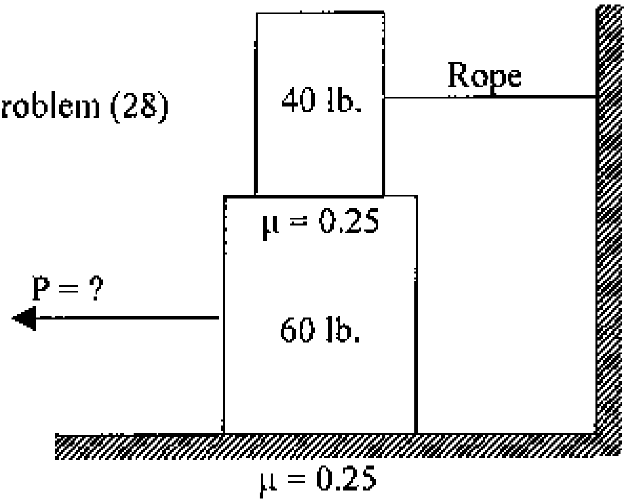
Problem (27): In the figure, the top block weighs 4 lb. and is acted upon by a horizontal force P. The block weighs 5 lb. The coefficient of friction between the two blocks is assumed to be 0.5, whereas the coefficient the lower block and the horizontal surface is assumed to be 0.2. What value of P will cause motion to impend?
Answer (27): P = 2 lb. (→)



Problem (27)

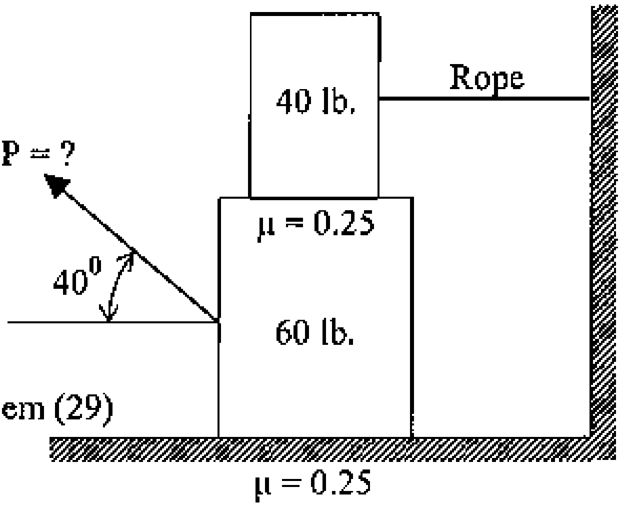
Problem (28): In the figure, the top block weighs 20 lb and the bottom block 60 lb. If the coefficient of friction is assumed to be 0.25 for all surfaces, what horizontal force P is needed to cause motion of the lower block to impend to the left? The rope is horizontal.
Answer (28): P = 35 lb. (←)

Problem (28)



Problem (29): In the figure, the top block weighs 20 lb and the bottom block 60 lb. If the coefficient of friction is assumed to be 0.25 for all surfaces, what inclined force P is needed to cause motion of the lower block to impend to the left? The rope is horizontal.

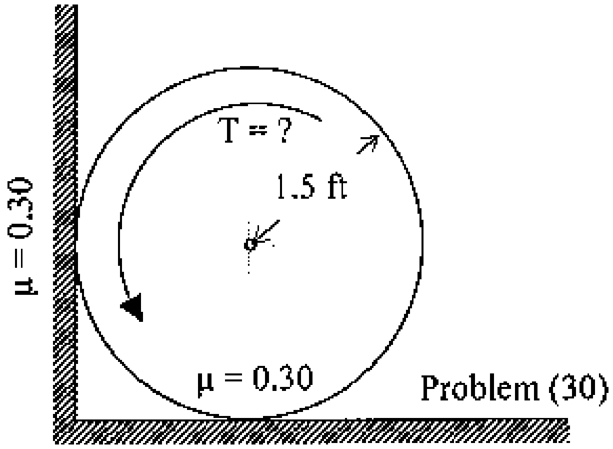
Answer (29): P = 37.7667406 lb.



Problem (29)

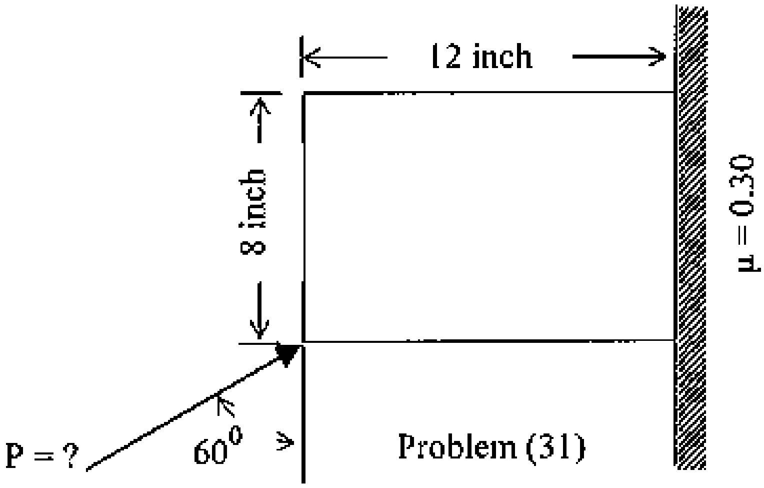
Problem (30): What couple/torque T is necessary to cause impending motion of the sphere of weight 30 lb and diameter 3 ft? Assume that the coefficient of friction for all surfaces is 0.30.

Answer (30): T = 16.10091743 ft-lb. (counterclockwise)



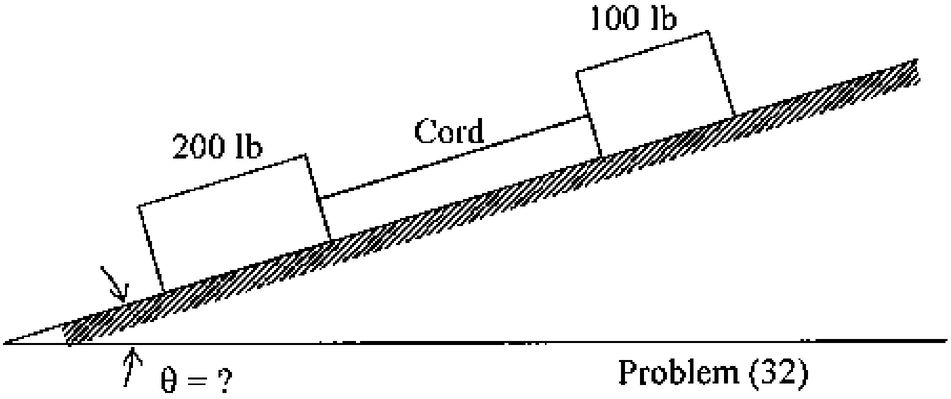
Problem (30)

Problem (31): A 20-lb block rests against a vertical wall for which the coefficient of friction is 0.30. A force is applied as shown. Determine the range of values that P may assume without causing motion to impend?
Answer (31): P = 19.75980762 lb. to 20 lb.



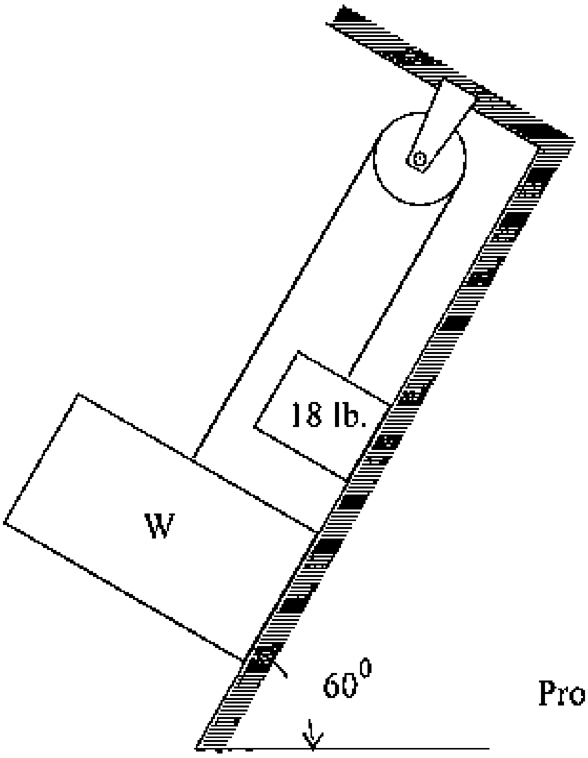
Problem (31)

Problem (32): A 200-lb block is connected to a 100-lb block by means of a cord parallel to the inclined plane. The coefficient of friction between the plane and the 100-lb block is 0.10. The coefficient is friction between the plane and the 200-lb block is 0.4. At what angle θ does motion impend down the plane?
Answer (32): $\theta = 16.69924423^0$



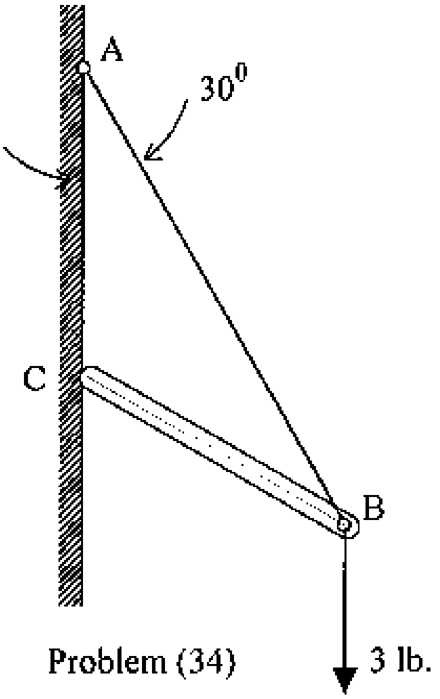
Problem (32)

Problem (33): What weight W will cause impending motion of the system. The coefficient of friction for all surfaces is 0.40. The pulley is frictionless and the cords are parallel to the 60^0 plane.
Answer (33): W = 5.40519923 lb.



Problem (33)

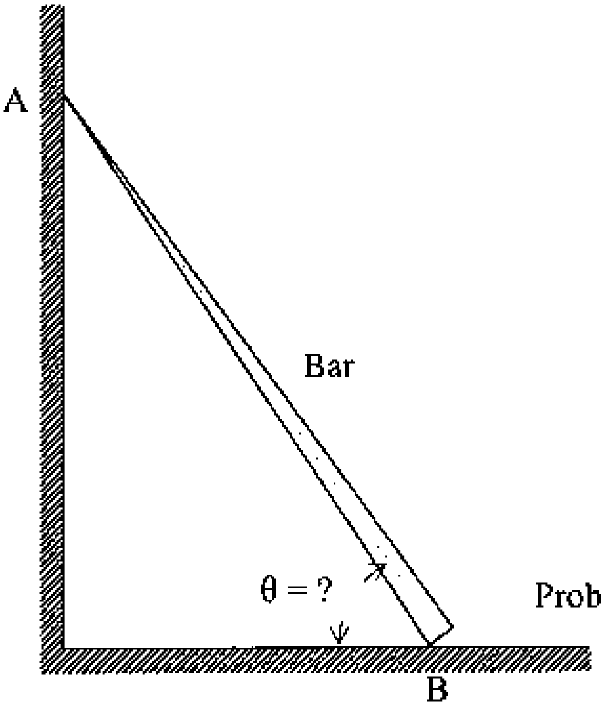
Problem (34): A weight of 3 lb. is held by the cord AB and the bar BC. Determine the necessary coefficient of friction (μ) between the bar and the vertical wall to ensure equilibrium.
Answer (34):



Problem (34)

Problem (35): A slender bar AB has its centroid one-third of the distance from the base B. if the co-efficient of friction for all surfaces is μ, determine the angle θ for equilibrium, that is, the value of θ below which the bar will slide.

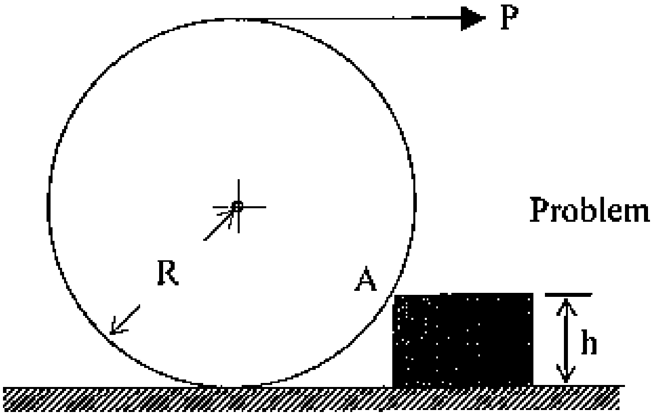
Answer (35): $\theta = \tan^{-1}\left(\dfrac{1-2\mu^2}{3\mu}\right)$



Problem (35)

Problem (36): In the figure, the horizontal force P, exerted by a rope wrapped around a homogeneous cylinder of weight W and radius R, is used to move the cylinder over an obstruction of height h. What must be the minimum coefficient of friction at A?

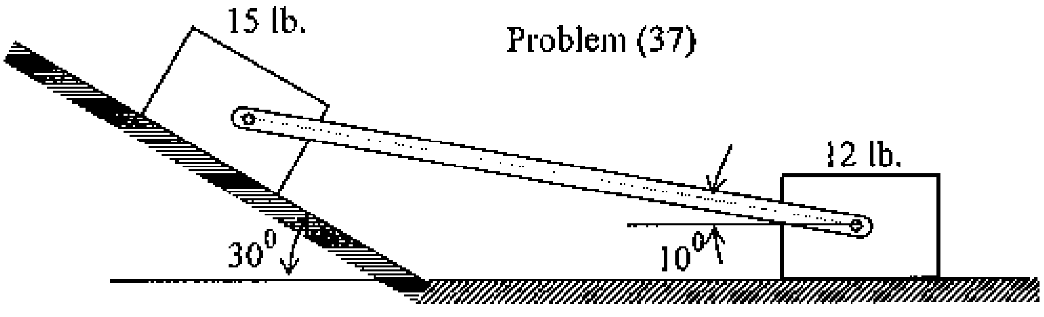
Answer (36): $\mu = \dfrac{2Rh}{2Rh^2 + h - h^3 - R}$



Problem (36)

Problem (37): The two blocks as shown in the figure are connected by a rigid weightless bar. What coefficient of friction is needed between the blocks and the surfaces to ensure equilibrium?

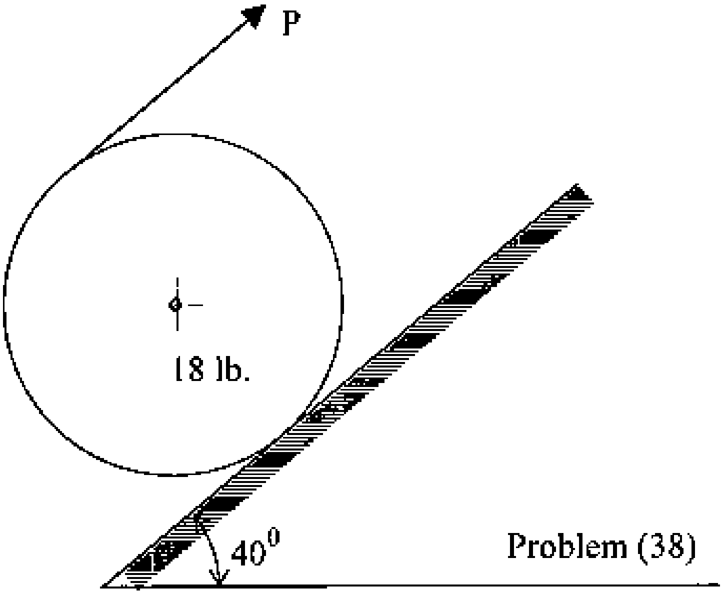
Answer (37): $\mu = 0.297565084$



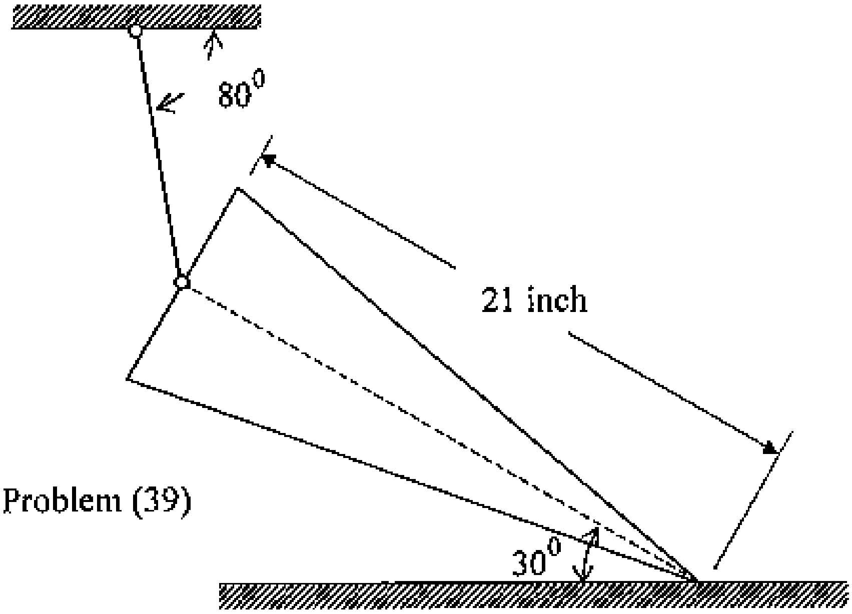
Problem (37)

Problem (38): The 18-lb homogeneous cylinder is acted upon by a force P parallel to the plane and exerted by means of a rope wrapped around the circumference. What coefficient of friction is necessary so that the cylinder will not rotate?

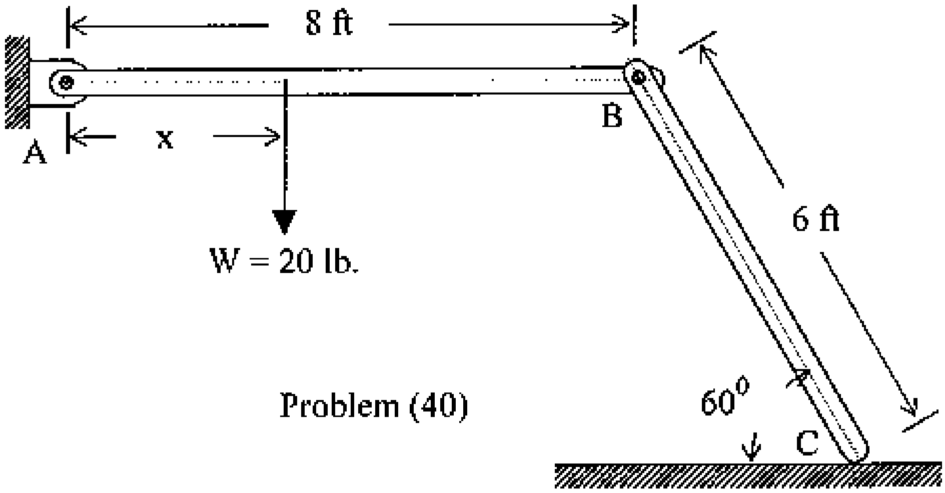
Answer (38): $\mu = 0.419549815$



Problem (39): A uniform isosceles triangular plate weighs 40 lb. The axis of symmetry makes an angle of 30^0 with the horizontal plane and is 21 inch long. Determine the necessary coefficient of friction to prevent slipping.
Answer (39): $\mu = 0.507713306$

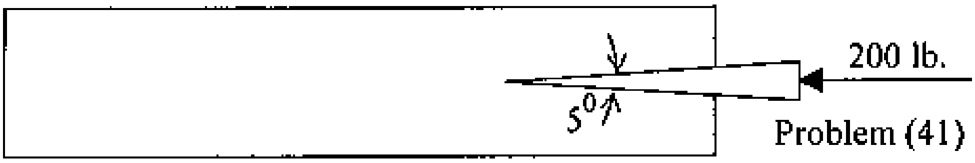


Problem (40): At what maximum value of x can the force W = 20 lb. be applied to maintain equilibrium if the coefficient of friction at the horizontal plane is 0.5? The horizontal bar AB weighs 10 lb and the bar BC weighs 8 lb.
Answer (40): x = 0.742562584 ft = 9.810751013 inch (from left)



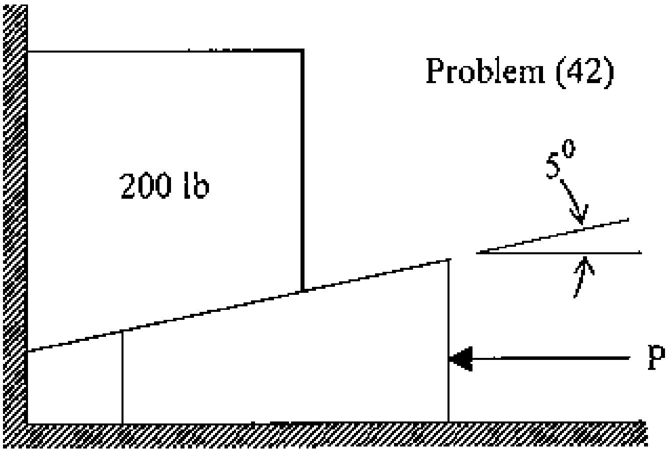
Problem (40)

Problem (41): A 5^0 wedge in a log is struck a blow equivalent to 200 lb. What is the splitting force N normal to the wedge? The coefficient of friction between the wedge and log is 0.2.
Answer (41): N = 2292.558536 lb.



Problem (41)

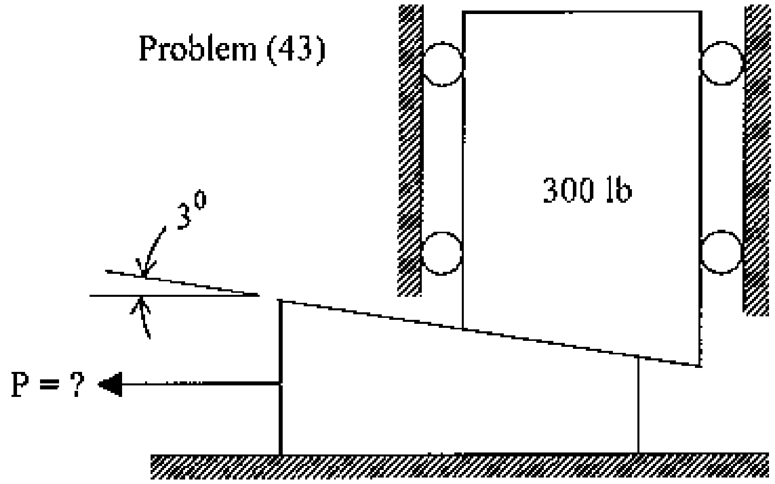
Problem (42): What horizontal force P is necessary to start the 5^0 wedge under the block of weight 200 lb? The floor is horizontal and the wall is vertical & smooth. The coefficient if friction for the wedge and the surfaces on which it reacts is 0.10.
Answer (42): P = 56.82267687 lb. (←)



Problem (42)

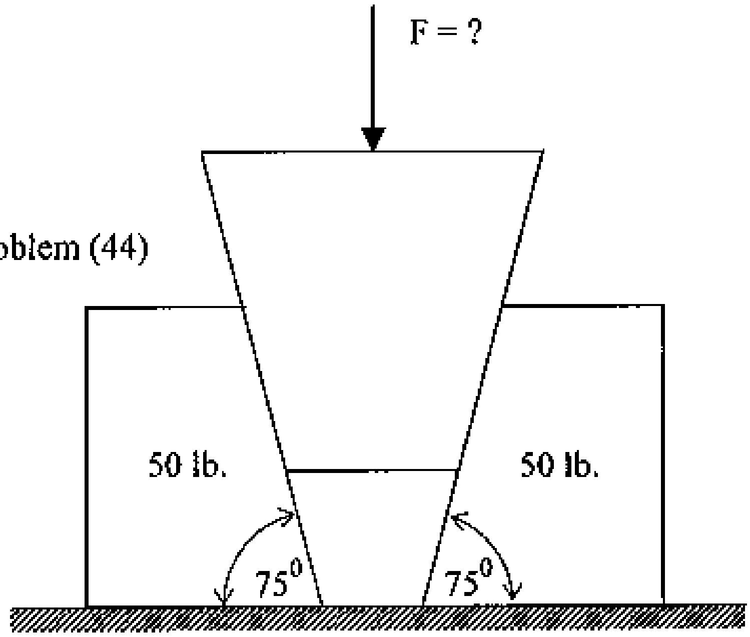
Problem (43): (a) Determine the horizontal force P necessary to withdraw the 30 wedge from under the 300-lb weight. The rollers can be considered to be frictionless. The coefficient of friction between the wedge and the surfaces on which it acts is 0.3. (b) What horizontal force P is needed to raise the block?
Answer (43): (a) 163.1279246 lb. (←); (b) 197.4110868 lb. (→)

Problem (43)



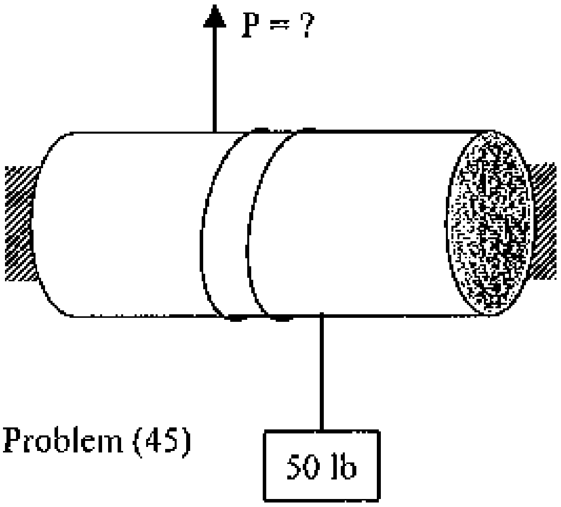
Problem (44): If the coefficient of friction for all sliding surfaces is 0.2, what value of F is required to move the 50-lb blocks? The force is vertical and the floor is horizontal.
Answer (44): P = 10.97415668 lb. (↓)

Problem (44)



Problem (45): A weight of 50 lb is suspended on a rope wrapped twice around a fixed horizontal cylinder. (a) What force P must be applied to the other end of the rope to prevent the weight from falling? (b)What force P would be necessary to raise the weight of 50 lb? The coefficient of friction between the rope and the cylinder is 0.25.

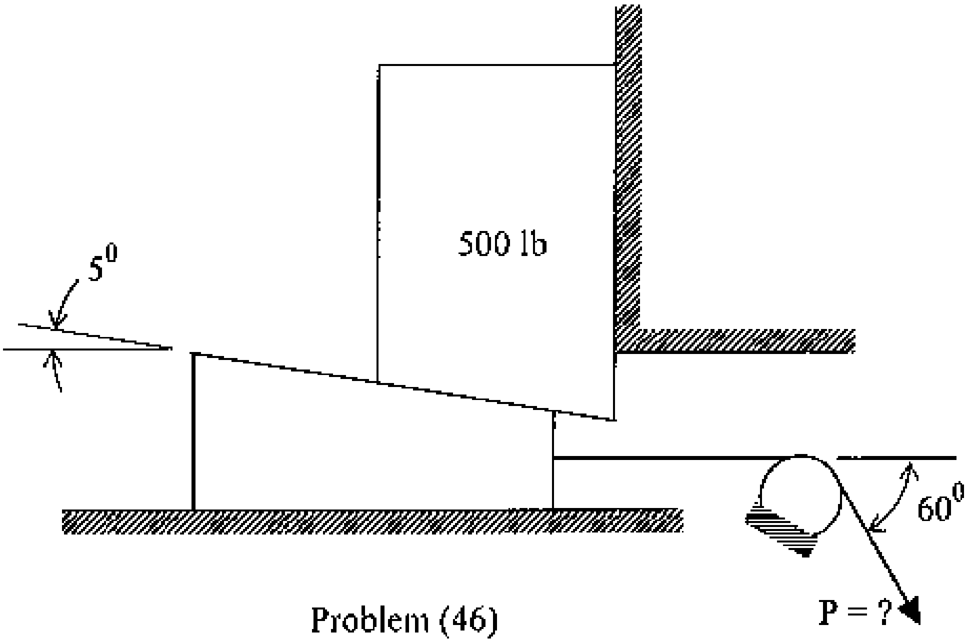
Answer (45): (a) 2.160695916 lb.; (b) 1157.034632 lb.



Problem (45)

Problem (46): Determine the force P necessary to raise a 500-lb block on top of a 5^0 wedge. The rope passes over a drum, the angle of wrap is 60^0, the coefficient of friction between the rope and the drum is 0.15. The coefficient of friction for all other sliding surface is 0.25.

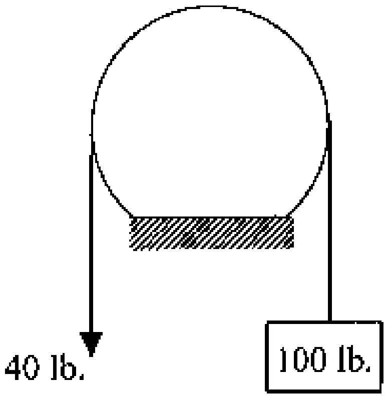
Answer (46): P = 380.9854461 lb.



Problem (46)

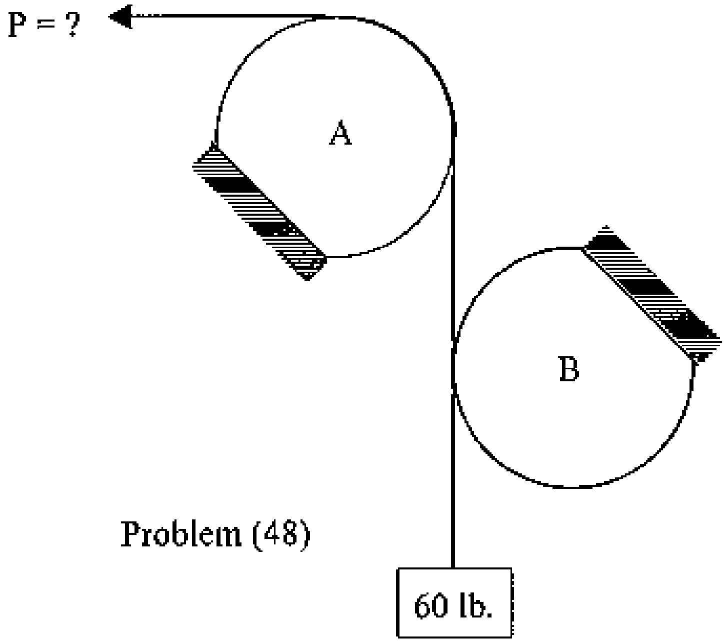
Problem (47): A weight 100 lb is kept from falling by a force of 40 lb applied at the end of a rope which has an angle of wrap of 180^0. What is the coefficient of friction between the rope and the fixed drum?

Answer (47): μ = 0.291664398



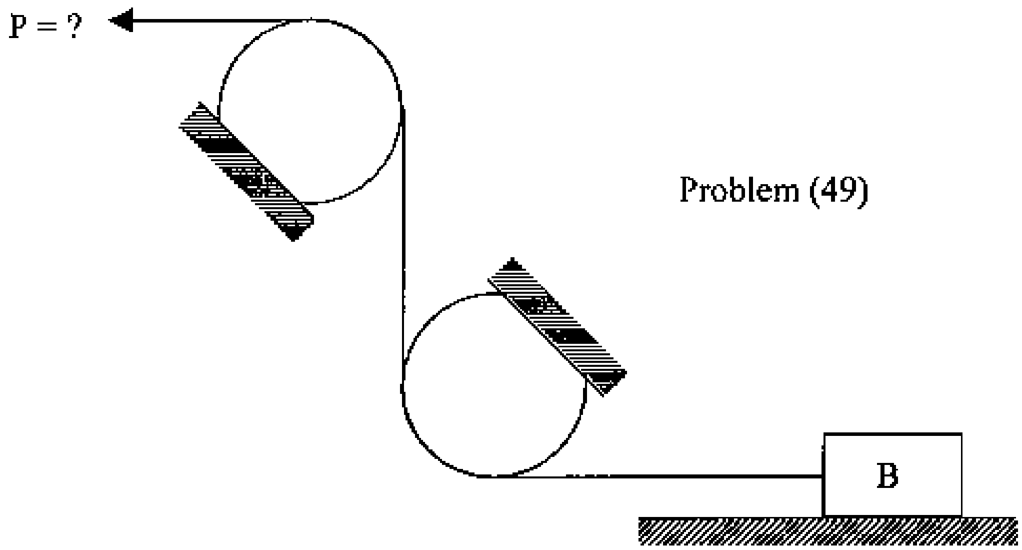
Problem (47)

Problem (48): A weight of 60 lb. is prevented from falling by a force P exerted on a rope which passes over two fixed drums. The rope makes a complete turn about B and one-quarter turn about A. Determine P if the coefficient of friction between the rope and both drums is 0.40.
Answer (48): P = 2.592835096 lb. (←)



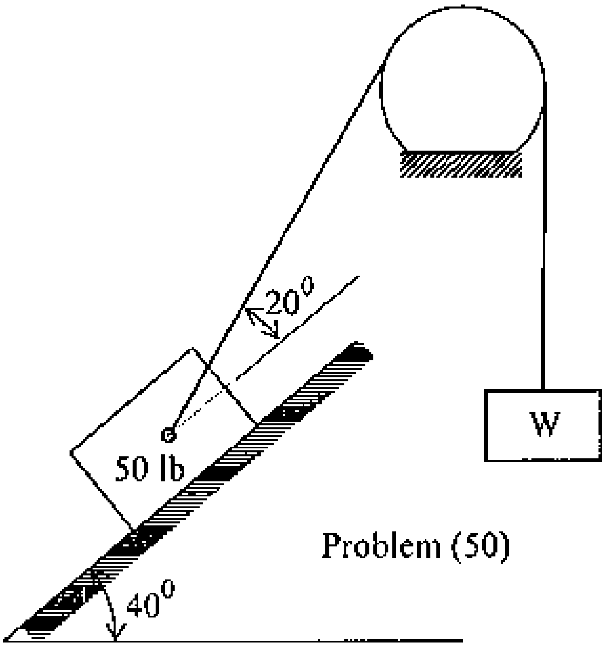
Problem (48)

Problem (49): Block B weighs 50 lb and rests on a horizontal plane for which the coefficient of friction is 0.3. A horizontal string is attached to B and passes over two friction drums, for which the coefficient of friction is 0.2. Determine the minimum value of the horizontal force P needed to move the block B.
Answer (49): P = 28.11684131 lb. (←)



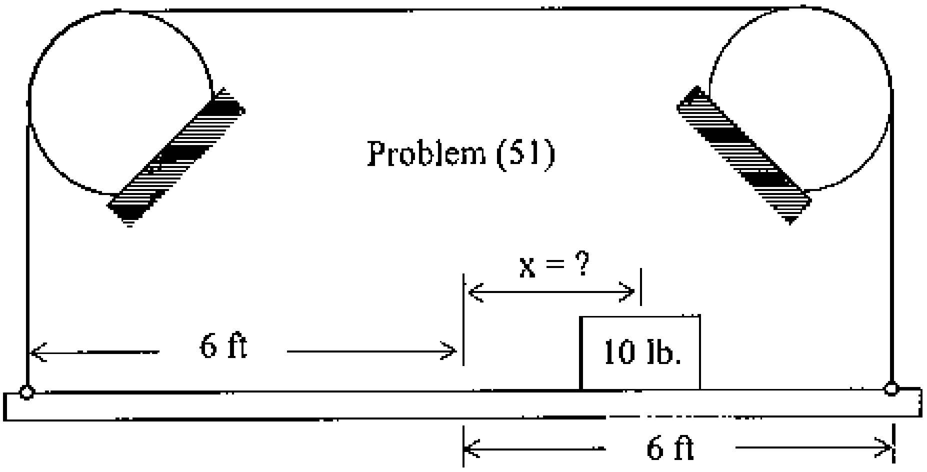
Problem (49)

Problem (50): Determine the range of values which W may have without disturbing equilibrium. The coefficient of friction is 0.20 between the block (weight 50 lb) and the inclined plane, as well as between the rope and the fixed drum over which the rope passes with an angle of wrap 150^0.
Answer (50): W = 16.64310875 lb. to 66.64614739 lb.

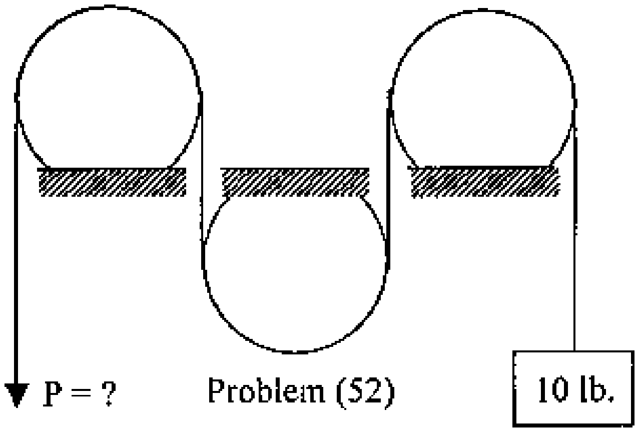


Problem (50)

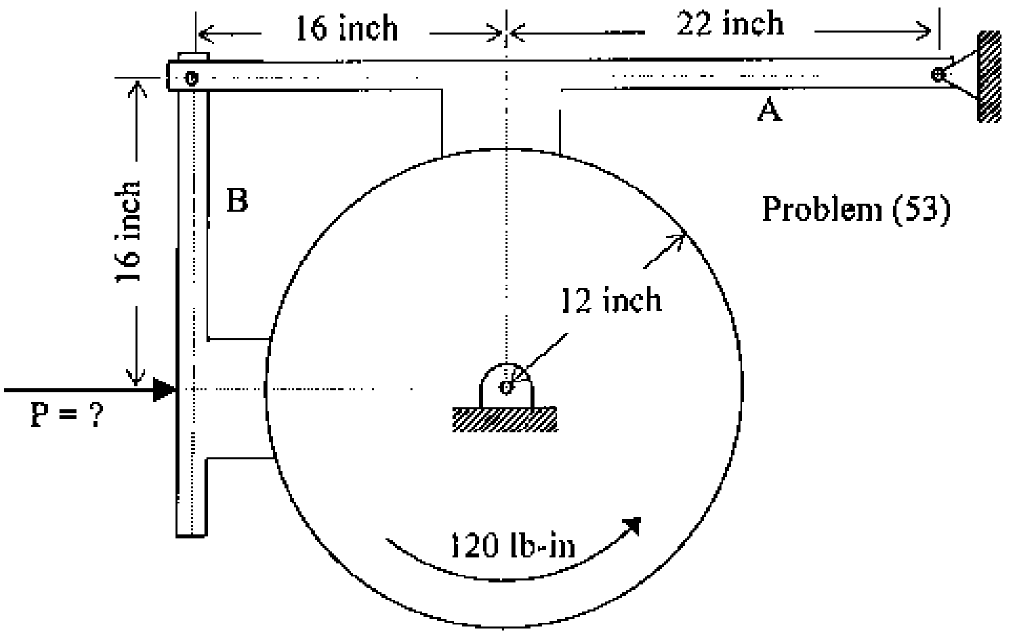
Problem (51): A platform, of negligible weight, is held horizontal by a rope fastened to each end and passing over fixed drums. If the coefficient of friction between the rope and each drum is 0.20. Determine how far from the center a 10-lb weight can be set without disturbing the balance.
Answer (51): 1.825297157 ft.



Problem (52): A weight 10 lb is about to fall. If the angle of wrap for each fixed drum is 180^0 and the coefficient of friction is 0.25, determine P for equilibrium.
Answer (52): 0.947802248 lb. to 105.5072407 lb.

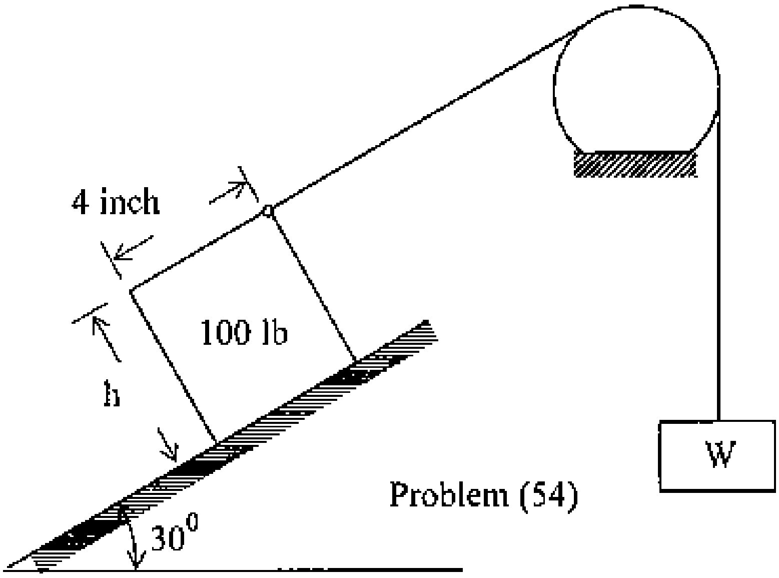


Problem (53): The drum is subjected to a 120 lb-in. counterclockwise moment. What horizontal force P is needed to forestall motion? The top member A is horizontal and B vertical. The coefficient of friction for two braking surfaces is 0.3.
Answer (53): P = 2402697495 lb. (→)



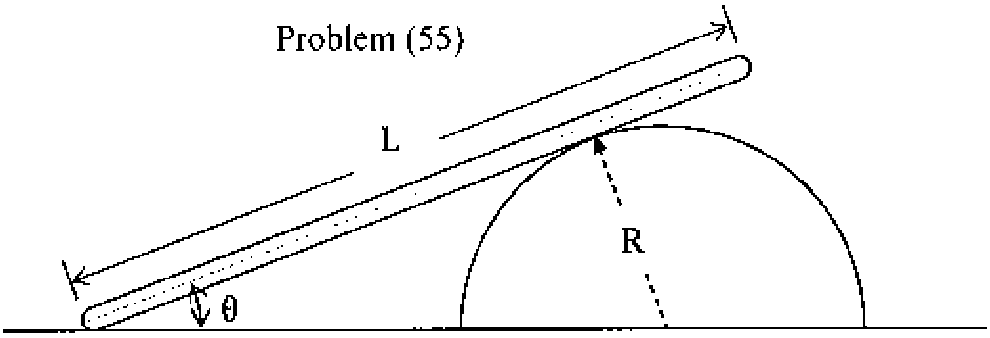
Problem (54): Determine the value of W and h so that the 100-lb block is on the verge of both sliding up the plane and tipping. The coefficient of friction between the rope and the fixed drum, as well as between the block and plane, is 0.2. The rope between the weight and drum is parallel to the plane.

Answer (54): W = 102.3444349 lb. (↓) & h = 4.09269852 inch.



Problem (55): A uniform bar of length L rests with one end on a horizontal floor and with another point on a half-cylinder of radius R. The coefficient of friction on the ground and on the cylinder are both equal to the same value μ. Derive the relation between the angle θ and the coefficient of friction μ for impending slip.

Answer (55): $\frac{1+\mu^2}{\mu}\sin^2\theta = \frac{2R}{L}$



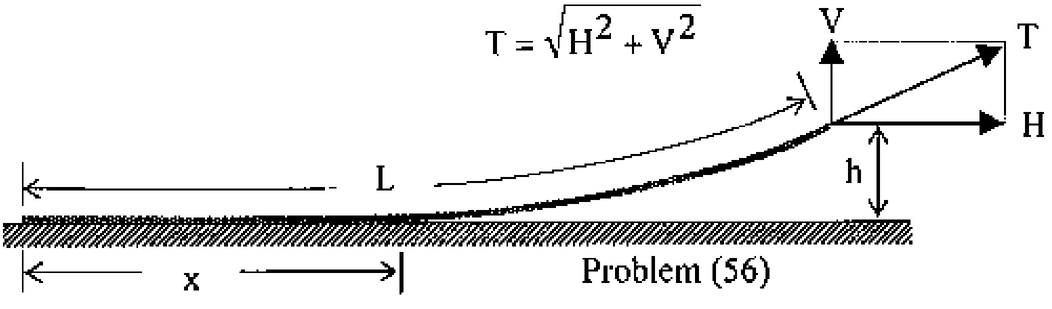
Problem (56): A rope of length L and weight W = wL (w is weight per unit length) is to be dragged along a rough floor with a coefficient of friction μ. It can be lifted at the pulling end to a height h above ground. In the calculation, the difference in length between the curved portion of the rope and its horizontal projection is to be neglected.

(a) What horizontal pull H is required to just drag the rope, and what is the distance x in this condition?

(b) What is the vertical component V of the pull at the pulling end in that condition?

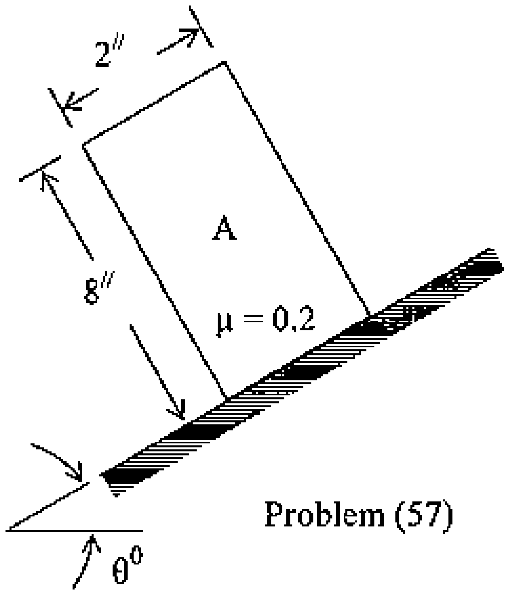
(c) Calculate the total pull T (the resultant of the horizontal and vertical components) for the case that L = 100 ft, w = 2 kip/ft, μ = 0.50 and h = 0.

(d) Calculate the total pull T (the resultant of the horizontal and vertical components) for the case that L = 100 ft, w = 2 kip/ft, μ = 0.50 and h = 3 ft.



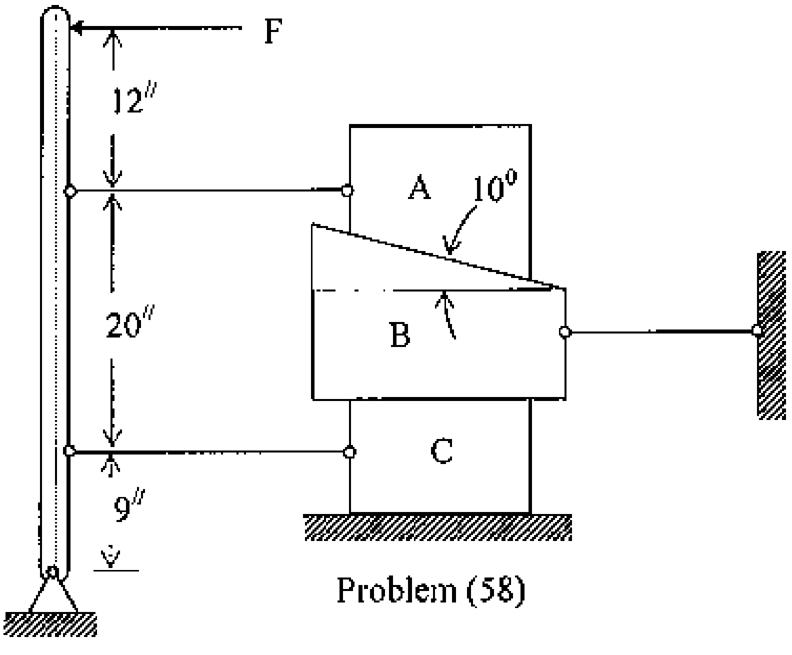
Answer (56): (a) $x = L + \mu h - \sqrt{2\mu hL + \mu^2 h^2}$ & H = μwL; (b) V = w(L-x); (c) T = 100 kip & (d) T = 89.91469396 kip.

Problem (57): The angle θ is gradually increased until the motion of A impends. Does the body tip over or slide?
Answer (57): The body will slide first.



Problem (57)

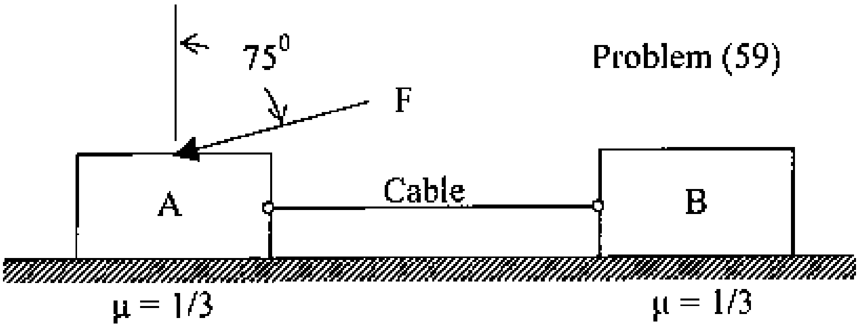
Problem (58): A, B & C each weighs 500 lb. and coefficient of friction (μ) 0.25 for all surfaces. Find the force F that is needed if counterclockwise rotation of the lever is impending.
Answer (58): F = 294.8803578 lb. (←)



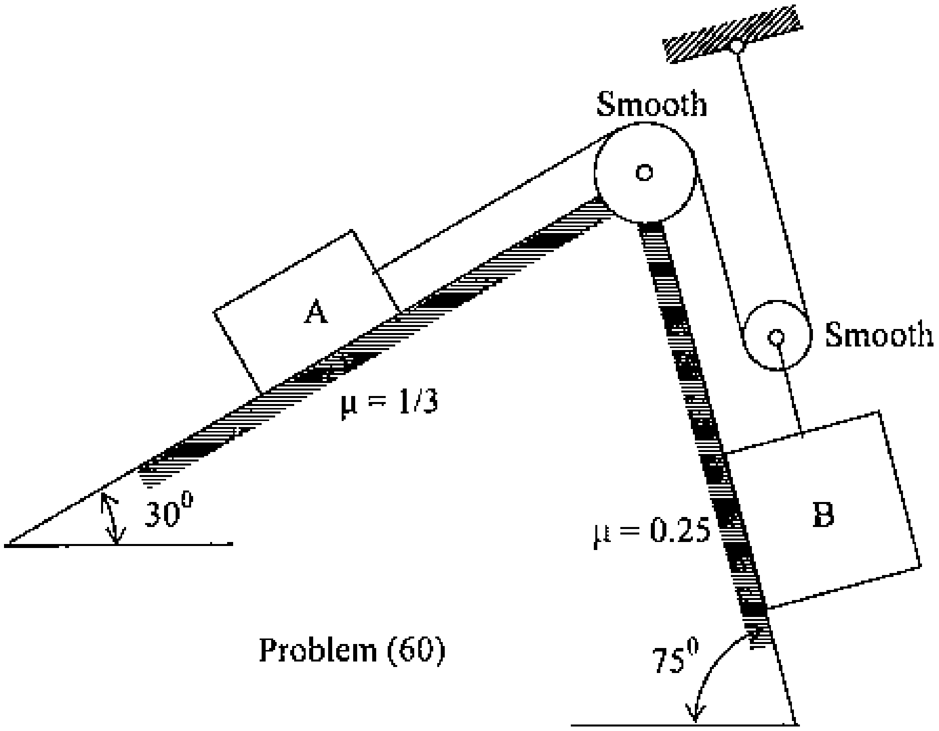
Problem (58)

Problem (59): A block weighing 300 lb., is acted upon by a force F whose direction is at 75^0 with the vertical. A weightless cable connects A to another

block B, which weighs 500 lb. (a) What force F will cause impending motion toward the left? (b) What is the tension in the cable?
Answer (59): (a) F = 303.1499056 lb. & (b) 166.6666667 lb.



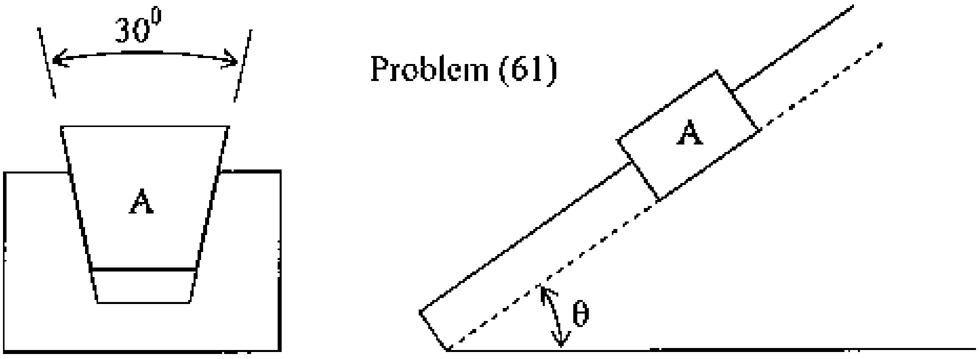
Problem (60): Given, weight of body A = 400 lb. (a) If the body B is on the point of moving downward, determine the tension in the cable and the weight of B & (b) If the body B is on the point of moving upward, determine the tension in the cable and the weight of B.
Answer (60): Tension in the cable = 315.4700538 lb. & weight of B = 700.0947183 lb.; (b) Tension in the cable = 84.52994616 lb. & weight of B = 164.0353919 lb.



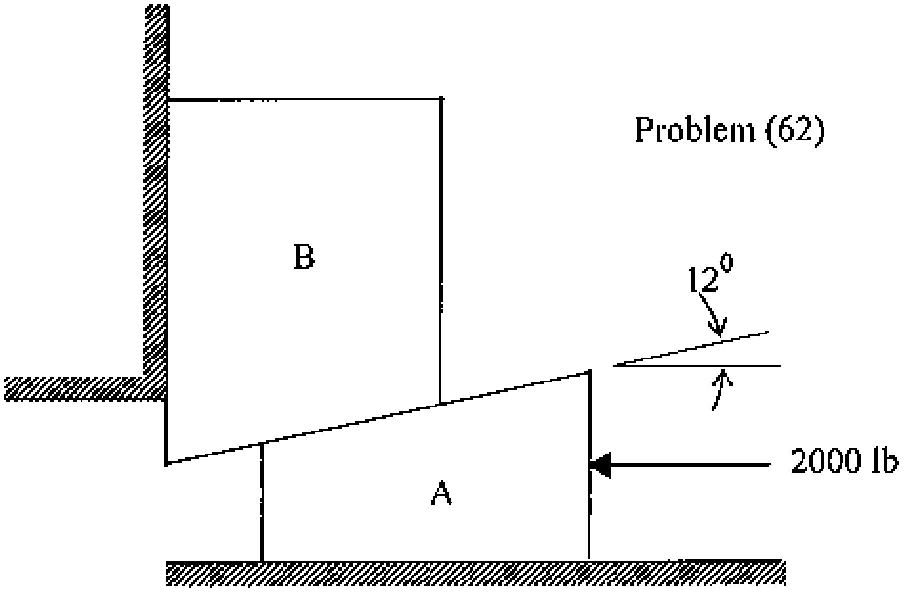
Problem (60)

Problem (61): A body A rests in a 30^0 groove. If coefficient of friction (μ) = 0.3, at what angle θ will motion of the body impend?

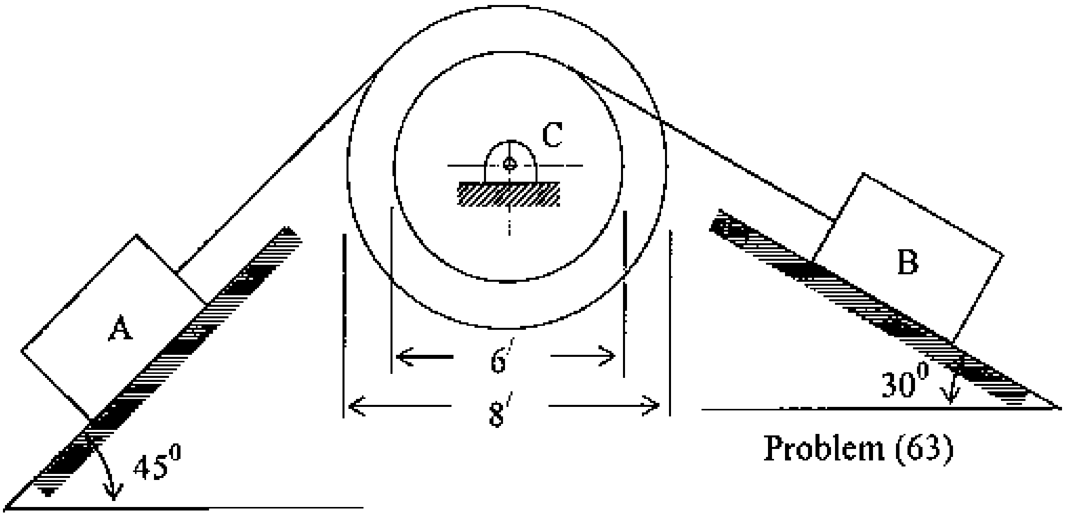
Answer (61): $\theta = 49.21466951^0$



Problem (62): If wedge A is weightless, what load W_B may be raised by a force 2000 lb? Coefficient of friction for all slipping surfaces (μ) = 0.25.
Answer (62): $W_B = 2401.811205$ lb. (↓)



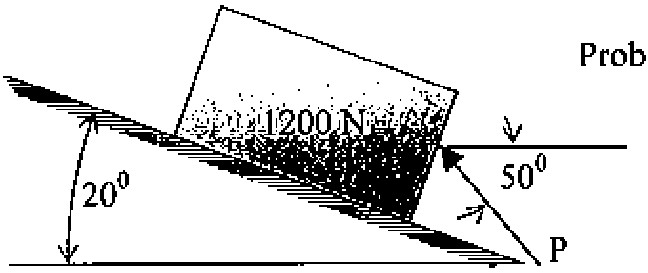
Problem (63): Given, W_A = 2000 lb., μ_A = 1/3 & μ_B = 0.15. For impending clockwise motion of the pulley, what is W_B? The bearing at C is smooth.
Answer (63): $W_B = 3821.205413$ lb.



Problem (63)

Problem (64): Determine the magnitude & direction of actual friction force on the 1200 N block, if P = 800 N & (ii) P = 300 N. the coefficients of static & kinetics friction are 0.25 & 0.20 respectively. The force P makes an angle of 50° with the horizontal.

Answer (64):

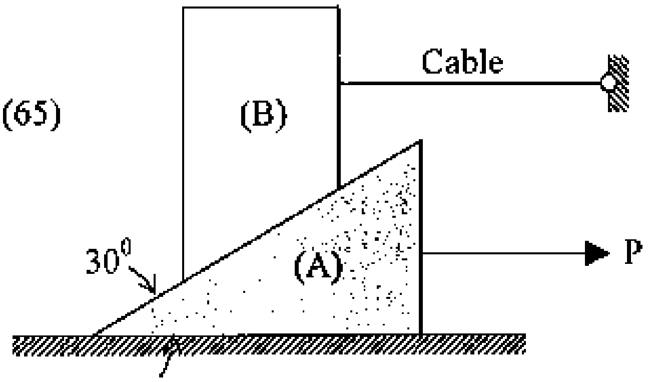


Problem (64)

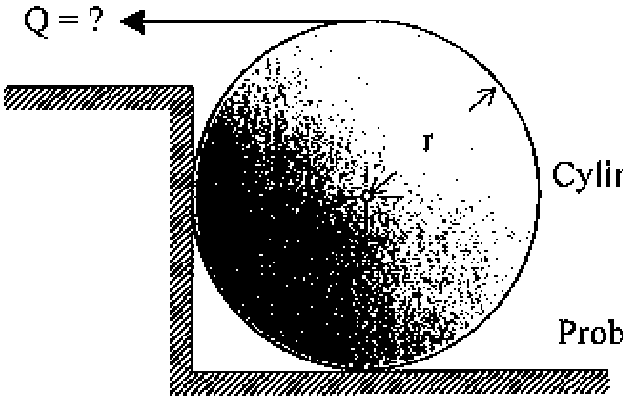
Problem (65): Two blocks A & B, weigh 600 N & 300 N, respectively. The coefficient of static friction between block A, and the horizontal plane is 0.3 while it is 0.2 between the blocks A & B. determine the magnitude of force P that will cause the block A to have impending motion towards right.

Answer (65): P = 317.6974553 N. (→)

Problem (65)

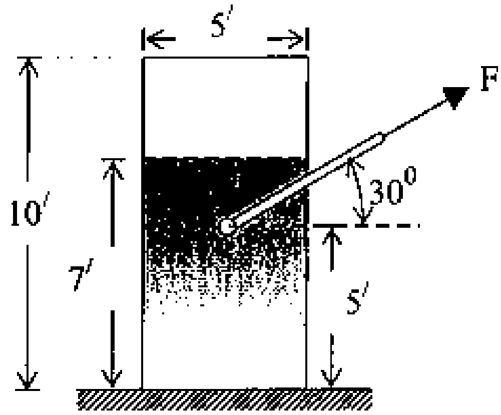


Problem (66): A Cylindrical oil drum is pulled by a force Q. the drum weighs 1500 N. the value of coefficient of static friction for all the surfaces is 0.5. Calculate the magnitude of the force Q when the drum just starts spinning.
Answer (66): Q = 1125 N (←).



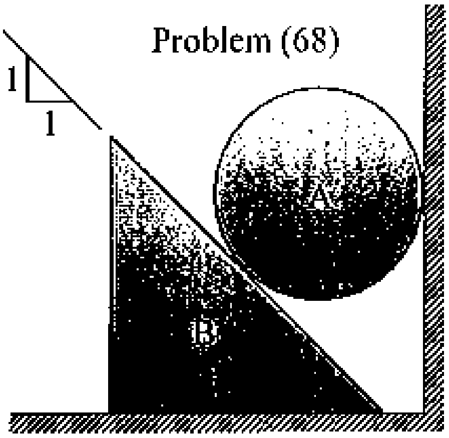
Problem (66)

Problem (67): A tank ($10' \times 5' \times 3'$) weighing 500 lb. containing water (unit weight = 62.4 lb./ft^3) is to be moved by a force F on each of two parallel connecting rods. If $\mu_s = 0.2$, what force is required to move this tank along the floor.
Answer (67): 1460.003013 lb. on two ropes (= 730.0015064 lb. on each rope).



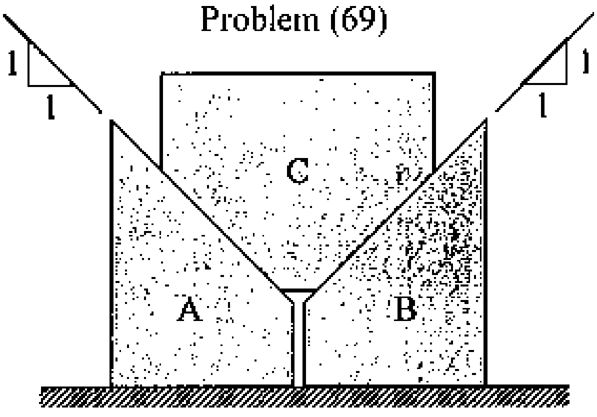
Problem (67)

Problem (68): A cylinder A weighing 100 lb. rests on a block B weighing 500 lb. The cylinder has a diameter if 1 foot. If $\mu_s = 0.5$ for all contact surfaces, can the system stay in equilibrium?
Answer (68): the arrangement will be in equilibrium (cylinder A move downward & block B move towards left), because actual frictional force between block B & base 28.57142857 lb. which is less than limiting frictional force 292.8571429 lb.


Problem (68)

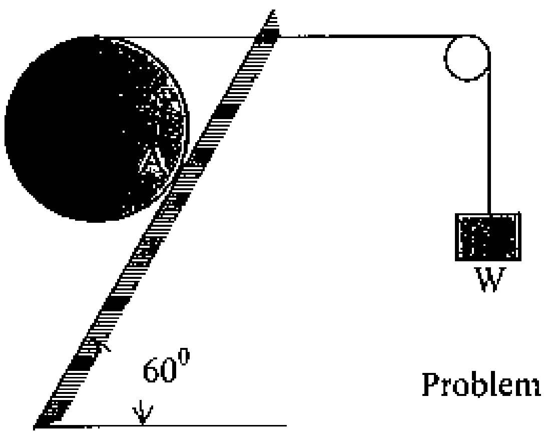
Problem (69): Block C weighs 200 lb. and identical blocks A & B weigh 150 lb. each. If μ_s = 0.5 for all contact surfaces, can the arrangement remain in equilibrium.

Answer (69): the arrangement will not be in equilibrium (block C move downward, block A move towards left & block B move towards right), because actual frictional force between block A (or B) & base 66.66667 lb. which exceeds limiting frictional force 50 lb.


Problem (69)

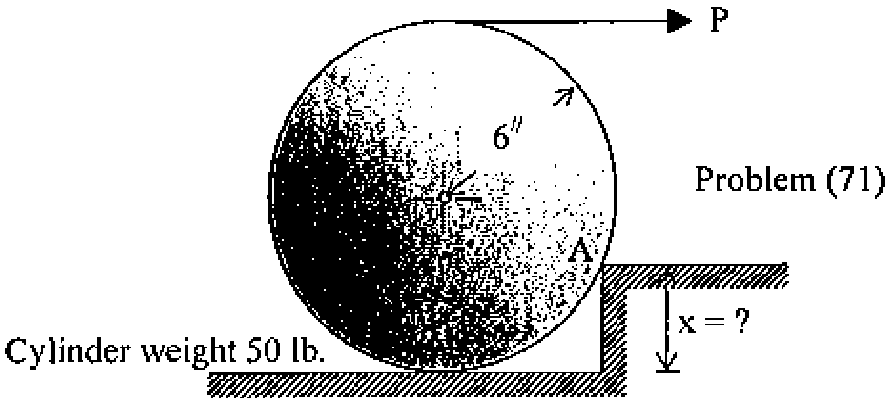
Problem (70): If there is a impending slippage at A, what is the friction co-efficient? Cylinder weight = 200 lb.

Answer (70): μ_s = 0.346410161.



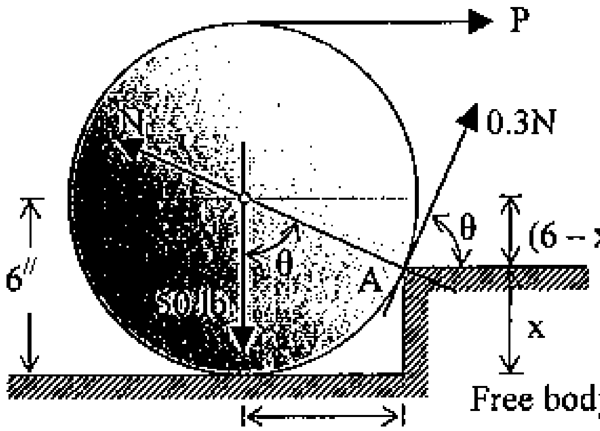
Problem (70)

Problem (71): What is the height 'x' of a step so that the force P will roll the 50 lb. cylinder over the step at the same time that there is impending slippage at A, take $\mu_s = 0.30$.



Solution (71): $\sum M_A = 0$ (clockwise +ve); $P \times (12 - x) = 50 \times \sqrt{12x - x^2}$

$$\Rightarrow P = \frac{50\sqrt{12x - x^2}}{(12 - x)}$$



Free body of cylinder

$$\sqrt{6^2 - (6 - x)^2} = \sqrt{12x - x^2}$$

$$\sin\theta = \frac{\sqrt{12x - x^2}}{6}; \cos\theta = \frac{(6 - x)}{6} \ \& \ \tan\theta = \frac{\sqrt{12x - x^2}}{(6 - x)}$$

$\sum F_x = 0$ ($\rightarrow$ +ve); $0.3N\cos\theta - N\sin\theta + P = 0$

$$\Rightarrow P = N\sin\theta - 0.3N\cos\theta = N(\sin\theta - 0.3\cos\theta)$$

$$\Rightarrow P = N\left[\frac{\sqrt{12x - x^2}}{6} - 0.3\frac{(6 - x)}{6}\right]$$

$$\Rightarrow \frac{50\sqrt{12x - x^2}}{(12 - x)} = N\left[\frac{\sqrt{12x - x^2}}{6} - 0.3\frac{(6 - x)}{6}\right] \quad \text{--- --- --- (a)}$$

$\sum F_y = 0$ (↑ +ve); $-50 + N\cos\theta + 0.3N\sin\theta = 0$

$\Rightarrow 50 = N\cos\theta + 0.3N\sin\theta = N(\cos\theta + 0.3\sin\theta)$

$$\Rightarrow 50 = N\left[\frac{(6-x)}{6} + 0.3\frac{\sqrt{12x-x^2}}{6}\right] \qquad \text{--- --- --- (b)}$$

Divide equation (a) by equation (b).

$$\Rightarrow \frac{\dfrac{50\sqrt{12x-x^2}}{(12-x)}}{50} = \frac{N\left[\dfrac{\sqrt{12x-x^2}}{6} - 0.3\dfrac{(6-x)}{6}\right]}{N\left[\dfrac{(6-x)}{6} + 0.3\dfrac{\sqrt{12x-x^2}}{6}\right]}$$

$$\Rightarrow \frac{\sqrt{12x-x^2}}{(12-x)} = \frac{\left[\sqrt{12x-x^2} - 0.3(6-x)\right]}{\left[(6-x) + 0.3\sqrt{12x-x^2}\right]}$$

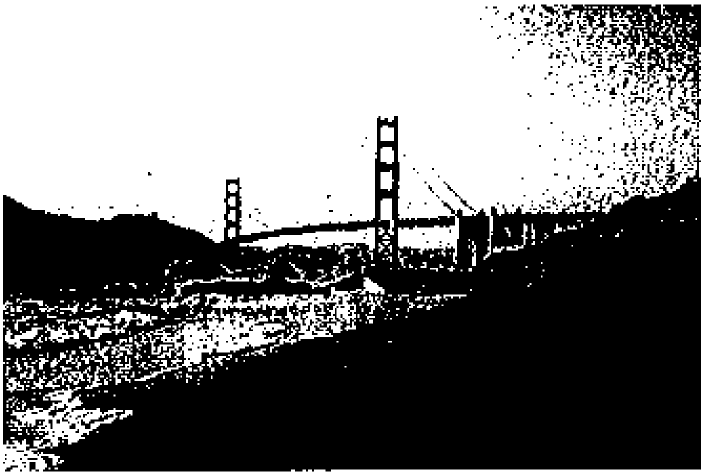
$\Rightarrow x = 0.990825688$ inch.

Answer (71): $x = 0.990825688$ inch.

Chapter Ten Flexible Cords (Cables)

A flexible cable in one in which the internal force is tension and is directed along the cable.

[Problems involving flexible cords are encountered in transmission lines and in suspension bridges. The load is usually assumed to be distributed uniformly in a horizontal direction or to be distributed uniformly along the cord.]



There are two common applications for the flexible cable: (1) the cable which is loaded uniformly in a horizontal direction, and (2) the cable which is loaded uniformly along its arc. An example of the first would be the main suspension cables for a suspension bridge. If we neglect the weight of the cables, in comparison with the weight of the roadway and vehicles, then the loading would be horizontal, as pictures if figure 1(a). An example of second application would be an electric transmission line, which hangs because of its own weight, and is hence uniformly loaded along the arc of the cable, as shown in figure 1(b).

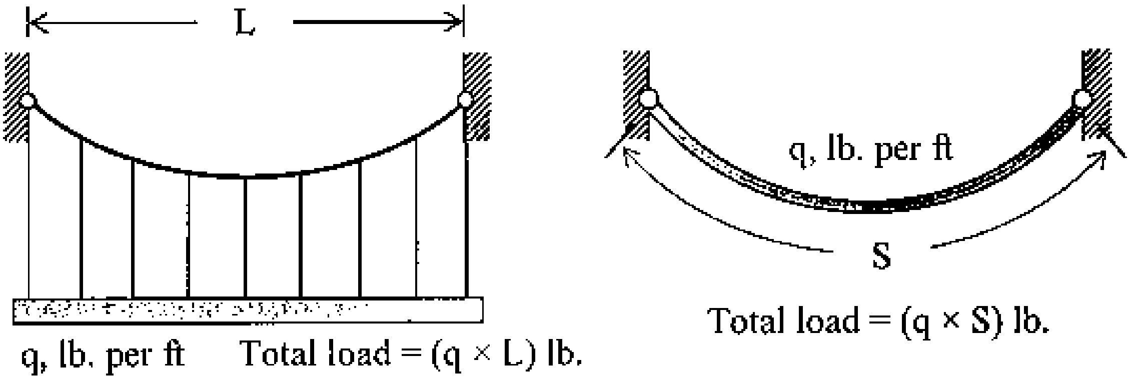


Figure 1: Flexible cords (a) parabola & (b) catenary

Parabolic Cords: A flexible chord loaded by uniform loading per unit of horizontal span length has the shape of parabola having small sag (consequently small slope).

q = load intensity along horizontal span length (uniform horizontal load).
L = the span (support to support distance).
S = length of the curve.
d = the sag (vertical distance of lowest point of curve from the support; if supports are at different levels, there are two values of the sag).
T = tensile force in the cable (a function of x).
H = horizontal component of tensile force T (constant along x, because the load has no x component) i.e., horizontal force at the low point.
V = vertical component of tensile force T (a function of x).

$$T = \sqrt{H^2 + V^2}$$

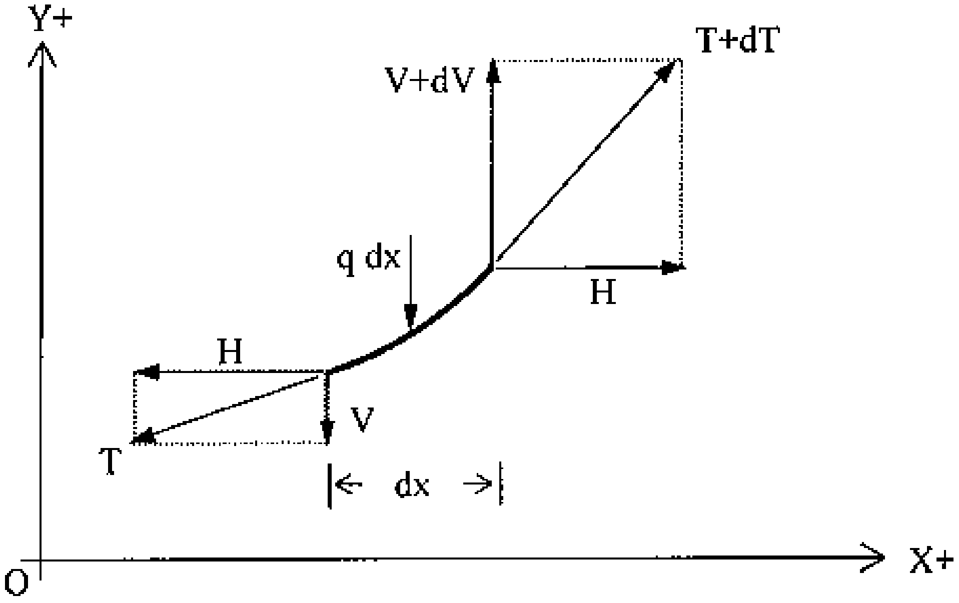


Figure 2: Equilibrium of an element of a parabolic cable

Summing the forces in horizontal direction; $\sum F_x = 0$; $H = H$

Summing the forces in vertical direction; $\sum F_y = 0$; $(V + dV) - V = q\,dx$

$$\frac{dV}{dx} = q$$

$$\frac{dy}{dx} = \frac{V}{H};\ V = H\frac{dy}{dx}$$

$$\frac{dV}{dx} = q = H\frac{d^2y}{dx^2}$$

$\frac{d^2y}{dx^2} = \frac{q}{H}$; Differential equation of the shape of the cable.

$$\frac{dy}{dx} = \frac{qx}{H} + C_1$$

$$y = \frac{qx^2}{2H} + C_1x + C_2$$

Boundary conditions: at $x = 0$, $y = 0$; $C_2 = 0$.

& at $x = 0$, $\frac{dy}{dx} = 0$; $C_1 = 0$.

$y = \frac{qx^2}{2H}$; Equation of the parabolic curve.

at $x = \frac{L}{2}$, $y = d$; $d = \frac{q(L/2)^2}{2H}$; $d = \frac{qL^2}{8H}$; This sag is seen to be proportional to the load & inversely proportional to the horizontal tensile force in the cable, as it should be.

$$\boxed{y = \frac{qx^2}{2H}}$$

The curve is always parabola, even if the two end supports are at unequal heights. In that case, the origin of co-ordinates, still at the lowest point of the curve, is no longer in the center of the span.

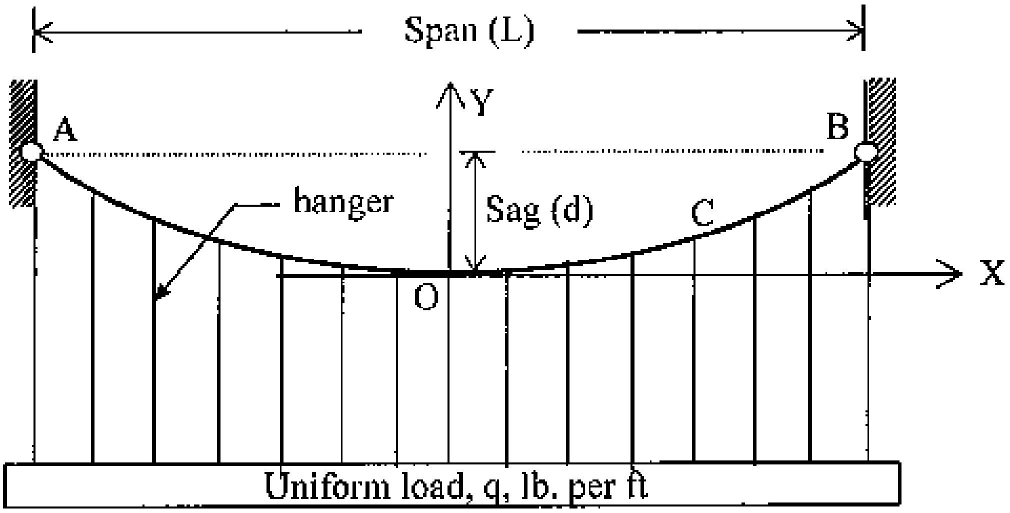


Figure 3(a): Flexible cord (parabolic shape)

Maximum vertical support reaction = $\frac{qL}{2}$

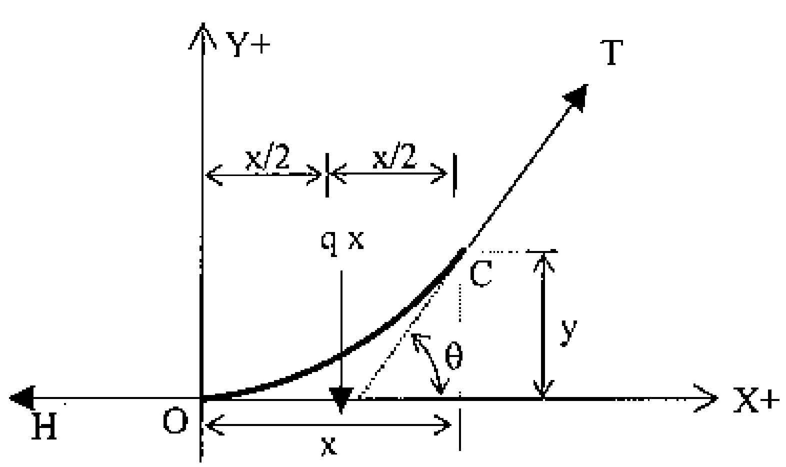


Figure 3(b): Equilibrium of an element of a parabolic cable

Summing the forces in horizontal & vertical directions

$\sum F_x = T\cos\theta - H = 0$

$T\cos\theta = H$

and $\sum F_y = T\sin\theta - qx = 0$

$T\sin\theta = qx$

Taking moments about point C gives

$$\sum M_c = 0; Hy - \frac{qx^2}{2} = 0 \Rightarrow y = \frac{qx^2}{2H}$$

$T = \sqrt{(T\cos\theta)^2 + (T\sin\theta)^2} = \sqrt{(H)^2 + (qx)^2}$; This gives the tension in cord at any abscissa x. Since the tension increases as x increases, the maximum tension occurs at the supports A & B.

Minimum tension (which occurs at the lowest point (at x = 0, y =0))

$= T_{min} = H$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

The slope of cord at any point C is, $\tan\theta = \frac{qx}{H}$

At the supports, x = L/2 & $\tan\theta_{max} = \frac{q(L/2)}{H} = \frac{qL}{2H}$

Substituting, $H = \frac{qL^2}{8d}$; $\tan\theta_{max} = \frac{4d}{L}$

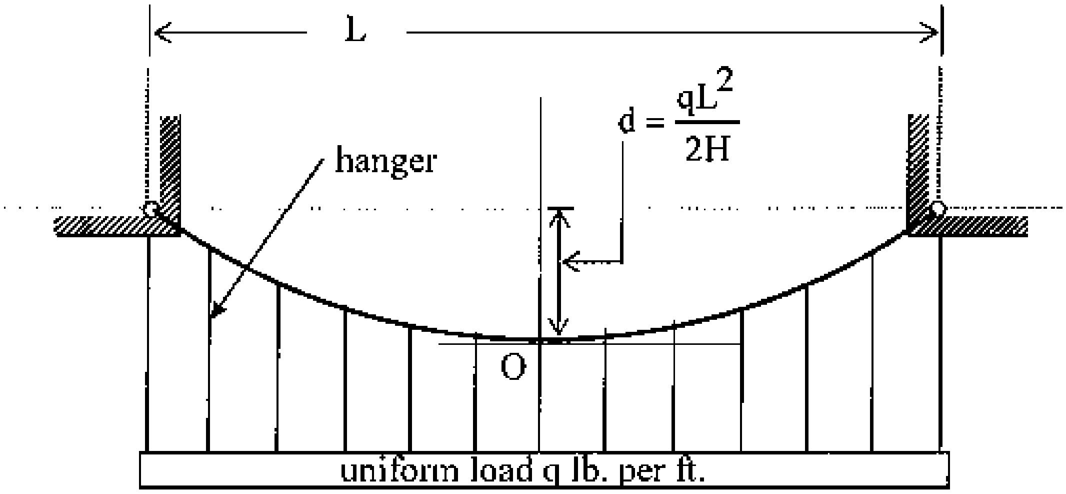


Figure 4(a): A parabolic cable uniformly loaded along its span

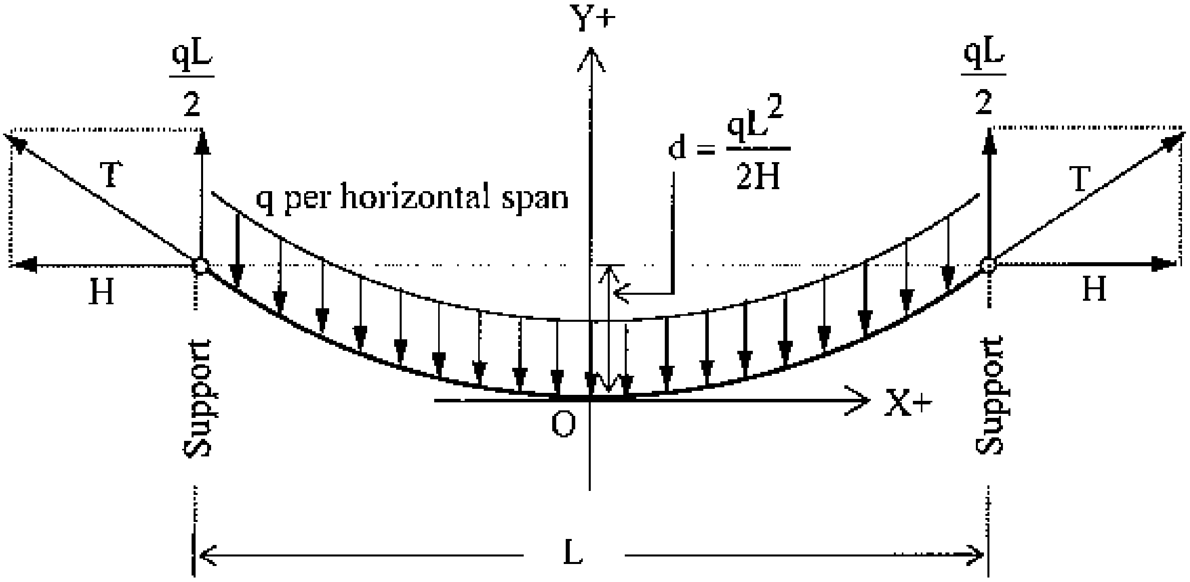


Figure 4(b): A parabolic cable uniformly loaded along its span

Length of a Parabolic Curve: In calculus, it is shown that the length of any curve is given by

$$S = \int \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx$$

$$\frac{dy}{dx} = \frac{qx}{H}$$

$$S = \int \left[1 + \left(\frac{qx}{H} \right)^2 \right]^{\frac{1}{2}} dx = \int_0^L \left[1 + \left(\frac{qx}{H} \right)^2 \right]^{\frac{1}{2}} = 2 \int_0^{L/2} \left[1 + \frac{q^2 x^2}{H^2} \right]^{\frac{1}{2}}$$

If we expand the integrand of the above equation in a binomial series, we get

$$\left[1 + \frac{q^2 x^2}{H^2} \right]^{\frac{1}{2}} = 1 + \frac{q^2 x^2}{2H^2} - \frac{q^4 x^4}{8H^4} + \frac{q^6 x^6}{16H^6} - \dots\dots\dots\dots$$

$$S = L + \frac{q^2 L^3}{24H^2} - \frac{q^4 L^5}{640H^4} + \frac{q^6 L^7}{7168H^6} - \dots\dots\dots\dots$$

Using $d = \frac{qL^2}{8H}$

$$S = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3} + \frac{256d^6}{7L^7} - \dots\dots\dots\dots$$

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

Where S_{max} is the length of parabolic cable between supports which are at the same level. Do not use this expression for supports at different level.

Catenary or Chainette: A flexible cord loaded by its own weight (uniformly loaded per unit of chord length) is called catenary or hanging-chain curve (comes from Latin *catena* (=chain)) having large sag (& consequently large slope).

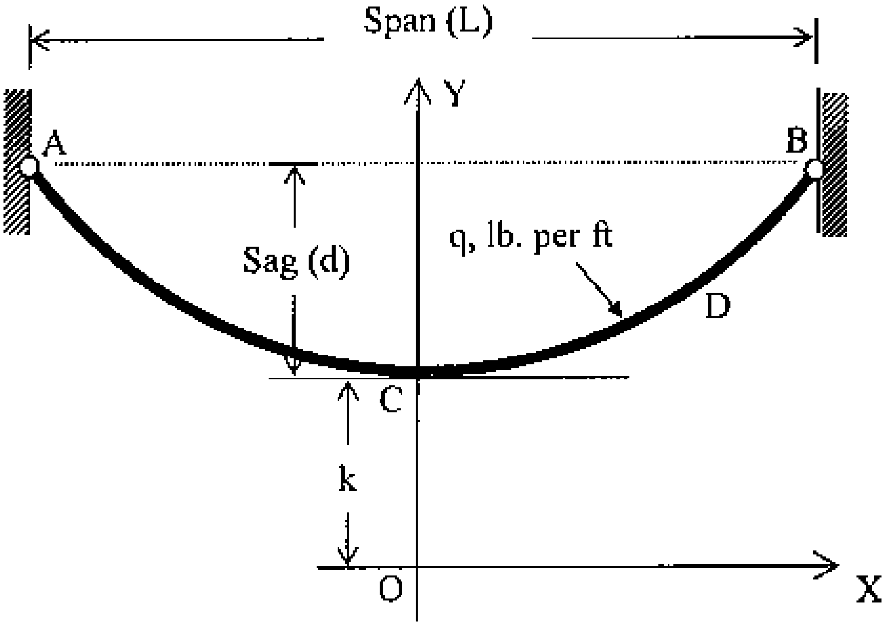


Figure 5(b): Flexible cord (catenary shape)

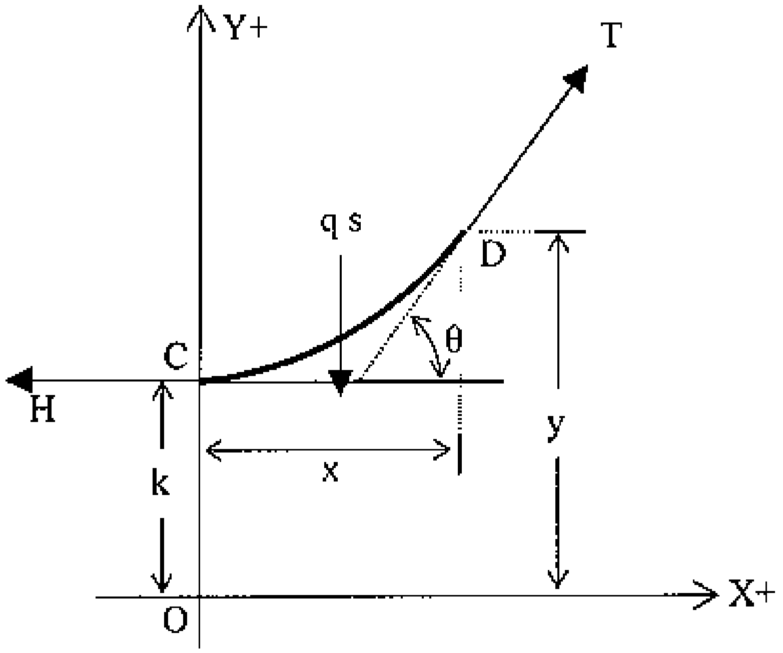


Figure 5(b): Equilibrium of an element of a catenary

q = load intensity along cable length (weight of unit length of cable)
L = the span (support to support distance).
S = length of the curve (note that length of the curve S is measured from the lowest point of the curve; if supports are at the same level, total length of the catenary is 2S).

d = the sag (vertical distance of lowest point of curve from the support; if supports are at different levels, there are two values of the sag).
T = tensile force in the cable (a function of s).
H = horizontal component of tensile force T (constant along x, because the load has no x component) i.e., horizontal force at the low point.
V = vertical component of tensile force T (a function of s).
k = H/q; the value of y at x = 0.

Summing the forces in horizontal & vertical direction.

$$\sum F_x = T\cos\theta - H = 0$$

$$T\cos\theta = H$$

and $\sum F_y = T\sin\theta - qs = 0$

$$T\sin\theta = qs$$

$$\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{qs}{H}$$

$$\frac{H}{q} = k$$

$$\frac{dy}{dx} = \tan\theta = \frac{qs}{H} = \frac{s}{(H/q)} = \frac{s}{k}$$

$$ds^2 = dx^2 + dy^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$

$$ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \left[1 + \left(\frac{s}{k}\right)^2\right]^{\frac{1}{2}} dx = \frac{(k^2 + s^2)^{\frac{1}{2}}}{k} dx$$

$$\frac{dx}{k} = \frac{ds}{(k^2 + s^2)^{\frac{1}{2}}}$$

$$\frac{x}{k} = \log_e\left[s + (k^2 + s^2)^{\frac{1}{2}}\right] - \log_e C_1$$

$$\frac{x}{k} = \log_e\left[\frac{s+(k^2+s^2)^{\frac{1}{2}}}{k}\right]$$

$$e^{\frac{x}{k}} = s+(k^2+s^2)^{\frac{1}{2}}$$

$$s = \frac{k}{2}\left[e^{\frac{x}{k}} - e^{-\frac{x}{k}}\right]$$

$s = k\sinh\left(\frac{x}{k}\right)$; Where s is the length of cord measured from the lowest point C.

If the cord is symmetric (points of support on the same level), the total cord length is 2s.

Substituting, $s = k\frac{dy}{dx}$

$$s = k\frac{dy}{dx} = k\sinh\left(\frac{x}{k}\right)$$

$$\frac{dy}{dx} = \sinh\left(\frac{x}{k}\right)$$

$$dy = \sinh\left(\frac{x}{k}\right)dx$$

$$y + C_2 = k\cosh\left(\frac{x}{k}\right)$$

$$y = k\cosh\left(\frac{x}{k}\right)$$

$$y = \frac{k}{2}\left[e^{\frac{x}{k}} + e^{-\frac{x}{k}}\right]$$

$$ds^2 = dx^2 + dy^2$$

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2$$

$$\frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}$$

$$ds = \left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy = \left[1+\left(\frac{k}{s}\right)^2\right]^{\frac{1}{2}} dy = \frac{(k^2+s^2)^{\frac{1}{2}}}{s} dy$$

$$dy = \frac{s\,ds}{(k^2+s^2)^{\frac{1}{2}}}$$

$$y + C_3 = (k^2+s^2)^{\frac{1}{2}}$$

$$y = (k^2+s^2)^{\frac{1}{2}}$$

$$T^2 = H^2 + V^2$$
$$T = \sqrt{H^2+V^2}$$
$H = qk$ & $V = qs$
$$T = \sqrt{(qk)^2+(qs)^2} = \sqrt{q^2(k^2+s^2)} = q\sqrt{(k^2+s^2)}$$
$$T = q\sqrt{(k^2+s^2)} = q\sqrt{y^2} = qy$$

This gives the tension in cord at any ordinate y. Since the tension increases as y increases, the maximum tension occurs at the supports A & B.

Minimum tension (which occurs at the lowest point (at x = 0 & y = k & s = 0)) $= T_{min} = H$

Maximum tension (which occurs at the supports (at x = L/2 & y = k+d)) is
$T_{max} = q\,(k+d)$

At supports, $y = (k^2+s^2)^{\frac{1}{2}} = k+d$

$$k+d = (k^2+s^2)^{\frac{1}{2}}$$

At the lowest point (point C) on the curve, y = k

The slope of cord at any point D is, $\tan 0 = \frac{qs}{H}$

At the supports, x = L/2, $y = (k^2+s^2)^{\frac{1}{2}} = k+d$

$$s = (k+d)^2 - k^2 = d^2 + 2kd$$

$$\tan\theta_{\max} = \frac{q(d^2 + 2kd)}{H}$$

Note: If origin is taken at the lowest point of the curve, then the equation of catenary becomes

$$\frac{y}{k} = \cosh\frac{x}{k} - 1$$

$$\frac{y}{(H/q)} = \cosh\frac{x}{(H/q)} - 1$$

$$\frac{q}{H}y = \cosh\frac{q}{H}y - 1$$

Comparison of Parabola & Catenary:

$y = \dfrac{qx^2}{2H}$; This equation of parabola can be written as:

$$\frac{q}{H}y = \frac{1}{2}\left(\frac{qx}{H}\right)^2$$

$\dfrac{q}{H}y = \cosh\dfrac{q}{H}y - 1$; This hyperbolic cosine of the catenary can be developed into a Taylor power series:

$$\frac{q}{H}y = \frac{1}{2}\left(\frac{qx}{H}\right)^2 + \frac{1}{24}\left(\frac{qx}{H}\right)^4 + \frac{1}{720}\left(\frac{qx}{H}\right)^6 + \ldots\ldots\ldots\ldots$$

The first term of this development is the same as the formula of the parabola, and the other terms represent the difference between the two, the catenary having larger deflection. The difference between the two becomes significant only for large values of $\frac{q}{H}x$, that is, for large sags in comparison to the span. The table below gives some figures which the reader should verify from the formula.

Table 1

$\frac{q}{H}x$	$\frac{\text{Sag}}{\text{Span}}$	Error
2	$\frac{1}{2}$	34%
$\frac{1}{2}$	$\frac{1}{8}$	2%
$\frac{1}{8}$	$\frac{1}{32}$	0.1%

Table 2: Difference between parabola & catenary

Parabola	Catenary
When the flexible cord is loaded uniformly along its horizontal span length, the cord becomes a parabolic shape. q = f(x)	When the flexible cord is loaded uniformly along its cord length, the cord becomes catenary shape. q = f(s)
$y = \frac{qx^2}{2H}$	$\frac{q}{H}y = \cosh\frac{q}{H}y - 1$
A catenary can be analyzed as a parabola When $\frac{\text{Sag}}{\text{Span}}$ ratio is $\leq 10\%$ (see table 1) & a parabola can be treated as a catenary for accurate results.	

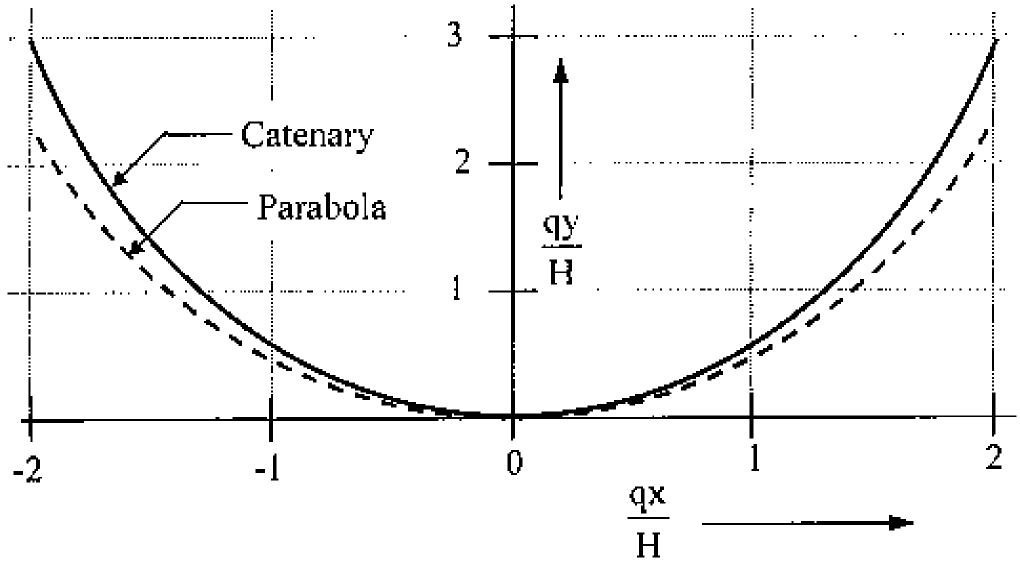


Figure 6: A parabola is a good approximation to a catenary for small sags

Problems

Problem (1): A cable is suspended with its ends at the same elevation and 200 feet apart. The load is uniformly distributed horizontally. When the sag is 5% of the span, the maximum tension is 2040 lb. What is the load in pounds per foot?

Solution (1): Span (L) = 200 ft.
Sag (d) = 5% of span = 5% of 200 = 20 feet.
Maximum tension (T_{max}) = 2040 lb.
Since, sag is 5% of span; the cable can be analyzed as a parabola.

Uniformly distributed load (per horizontal foot) = q lb/ft.
Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{q(200)^2}{8(20)} = 500q \text{ lb.}$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 2040 = \sqrt{(500q)^2 + \left(\frac{q \times 200}{2}\right)^2}$$

$$\Rightarrow q = 4.000799157$$

Answer (1): q = 4 lb/ft.

Problem (2): A cable weighing 2 lb. per ft. is 400 ft. long. The distance between the points of support, which are on a horizontal, is 398 ft. Compute the sag and maximum tension in the cable.

Solution (2): Span (L) = 398 ft.
Length of cable (S) = 400 ft
Uniformly distributed load (per horizontal foot), (q) = 2 lb/ft.

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow \ 400 = 398 + \frac{8d^2}{3(398)} - \frac{32d^4}{5(398)^3}$$

$$\Rightarrow d = 17.31653408 \text{ ft}$$

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{(2)(398)^2}{(8)(17.31653408)} = 2286.889502 \text{ lb.}$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow \ T_{max} = \sqrt{(2286.889502)^2 + \left(\frac{2 \times 398}{2}\right)^2}$$

$$\Rightarrow T_{max} = 2321.264223 \text{ lb.}$$

Answer (2): 17.31653408 ft; 2321.264223 lb.

Problem (3): A 3/8-inch copper wire may be safely loaded to a maximum tension of 12000 psi of cross-sectional area. For a sag of 20 ft., what may be the maximum safe span when the points of support are on the same level? Copper weighs 556 lb. per cft. Assume the curve to be parabolic.

Solution (3): Diameter of copper wire = 3/8 inch

Cross-sectional area of copper wire = $\frac{\pi(3/8)^2}{4}$ inch^2

$$= \frac{9\pi}{256} \text{ inch}^2 = \frac{\pi}{4096} \text{ ft}^2$$

Weight of copper wire (unit weight) = 556 lb. per cft

Uniform weight (q) = (Unit weight) × (cross-sectional area of wire)

$$q = 556 \frac{\text{lb}}{\text{ft}^3} \times \frac{\pi}{4096} \text{ ft}^2 = \frac{139\pi}{1024} \text{ lb. per ft.}$$

[cft = cubic feet]

Maximum tensile stress = 12000 psi

[psi = pound per squre inch]

Maximum tensile force (T_{max}) = (Maximum tensile stress) × (cross-sectional area of copper wire) = 12000 psi × $\frac{9\pi}{256}$ inch^2 = 421.875π lb.

Sag (d) = 20 feet

Minimum tension at lowest point = H

Span of cable = L

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{\left(\frac{139\pi}{1024}\right)(L)^2}{8(20)} = \frac{\pi}{655360} L^2 \text{ lb.}$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 421.875\pi = \sqrt{\left(\frac{\pi}{655360}L^2\right)^2 + \left(\frac{\frac{139\pi}{1024}\times L}{2}\right)^2}$$

$$\Rightarrow L = 702.9057994 \text{ ft.}$$

Answer (3): 702.9057994 ft.

Problem (4): A wire which weighs 1 lb, per ft. is to have a sag of 10% of the span. The points of support are on the same level. (a) If the maximum permissible tension in the wire is 1000 lb., what is the maximum permissible span? (b) What length of wire is needed?

Solution (4): Span (L) = 200 ft.
Sag (d) = 10% of span (L) = 0.1 (L)
Maximum tension (T_{max}) = 1000 lb.
Since, sag is 10% of span; the cable can be analyzed as a parabola.
Uniformly distributed load (per horizontal foot), (q) = 1 lb/ft.
Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{(1)(L)^2}{8(0.1L)} = 1.25L^2$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 1000 = \sqrt{(1.25L^2)^2 + \left(\frac{1\times L}{2}\right)^2}$$

$$\Rightarrow L = 742.7813527 \text{ ft}$$

Sag (d) = 0.1 × L = 0.1 × 742.7813527 ft = 74.27813527 ft
Length of cable = S feet

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 742.7813527 + \frac{8(74.27813527)^2}{3(742.7813527)} - \frac{32(74.27813527)^4}{5(742.7813527)^3}$$

$$\Rightarrow S_{max} = 762.1134754 \text{ ft.}$$

Answer (4): 742.7813527 ft; 762.1134754 ft.

Problem (5): A bridge cable is to be swung between two towers 1000 ft apart. The strength of the cable is 1200 kips. The loading on the cable is 300 lb per horizontal foot. Determine the necessary sag and the total length of the cable.

Solution (5): Uniform loading on the cable (q) = 300 lb per horizontal ft
Loading is given per horizontal feet; analyze the cable as a parabola.
Span (L) = 1000 feet
Sag = d
Maximum tension (T_{max}) = 1200 kip = 1200000 lb.

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 1200000 = \sqrt{H^2 + \left(\frac{300 \times 1000}{2}\right)^2}$$

$$\Rightarrow H = 1190588.09 \text{ lb.} = \text{Minimum tension at lowest point}$$

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow 1190588.09 = \frac{(300) \times (1000)^2}{8(d)}$$

$$\Rightarrow d = 31.49703942 \text{ feet} = \text{Sag}$$

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 1200 + \frac{8(74.16770791)^2}{3(1200)} - \frac{32(74.16770791)^4}{5(1200)^3}$$

$$\Rightarrow S_{max} = 1002.639204 \text{ feet} = \text{Length of cable}$$

Answer (5): 31.49703942 ft; 1002.639204 ft.

Problem (6): A cable, whose ends are anchored at the same elevation, carries a uniform horizontal load of 100 lb/ft. the maximum sag is to be 10 ft. determine the allowable span and length for a maximum tension of 3000 lb.

Solution (6): Sag (d) = 10 feet
Maximum tension (T_{max}) = 3000 lb.
Uniformly distributed load (per horizontal foot), (q) = 100 lb/ft.

Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{(100)(L)^2}{8(10)} = 1.25L^2$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 3000 = \sqrt{(1.25L^2)^2 + \left(\frac{100 \times L}{2}\right)^2}$$

$$\Rightarrow L = 41.5911304 \text{ ft}$$

Length of cable = S feet

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 41.5911304 + \frac{8(10)^2}{3(41.5911304\ )} - \frac{32(10)^4}{5(41.5911304)^3}$$

$$\Rightarrow S_{max} = 47.11318861 \text{ ft.}$$

Answer (6): 41.5911304 ft.; 47.11318861 ft.

Problem (7): A cable is strung between two towers 700 ft apart. It carries a uniform horizontal load of 400 lb/ft. The length of the cable is 430 ft. Determine the sag and maximum tension. The cable anchors are at the same elevation.
Answer (7): 90.57989762 ft; 304.5638317 ft.

Problem (8): A bridge cable is suspended between two towers at the same elevation. The span is to be 1500 ft, and the sag is 150 ft. the allowable design load for the tension at the towers is 6000 tons. Determine the allowable horizontal load per foot, and the required length of cable.

Solution (8): Uniform cable weight = q lb/ft
Span (L) = 1500 feet
Sag (d) = 150 feet

$$\frac{\text{sag}}{\text{span}} = \frac{150}{1500} = 0.01 = 10\%$$

Since, sag is 10% of span; the cable can be analyzed as a parabola.

Maximum tension (T_{max}) = 6000 ton

Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{(q)\times(1500)^2}{8(150)} = 1875\ q$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow\ 6000 = \sqrt{(1875\,q)^2 + \left(\frac{q\times 1500}{2}\right)^2}$$

$$\Rightarrow\ q = 2.971125411\ \text{ton/ft}$$

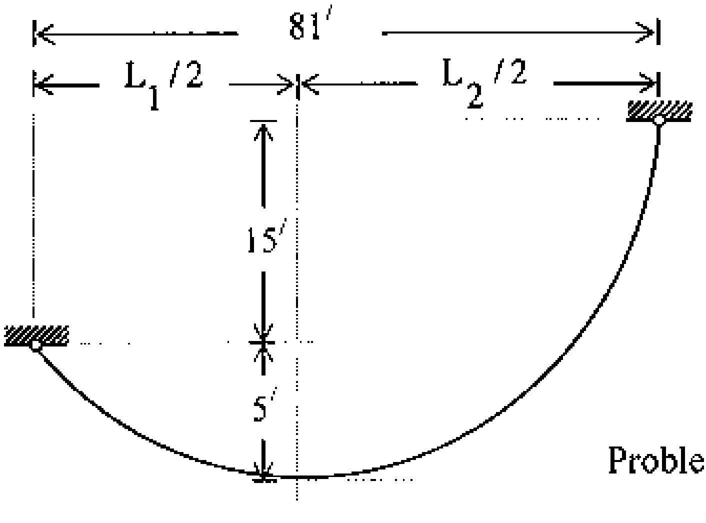
Length of cable = S feet

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

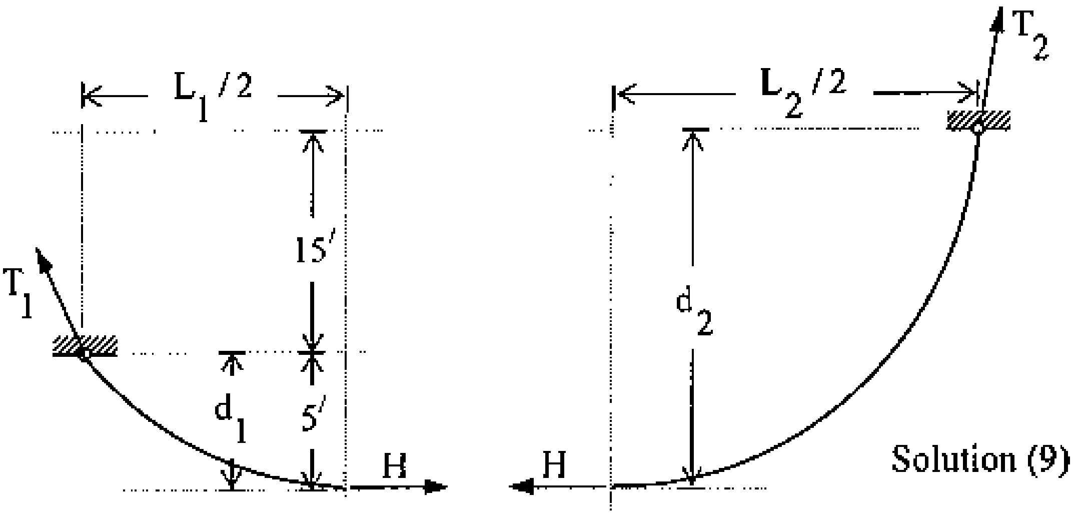
$$\Rightarrow\ S_{max} = 1500 + \frac{8(150)^2}{3(1500)} - \frac{32(150)^4}{5(1500)^3} = 1539.04\ \text{ft}$$

Answer (8): 2.971125411 tons/ft.; 1539.04 ft.

Problem (9): A cable is suspended between two points 81 horizontal feet apart. One of the cable anchors is 15 ft below the other, and the sag, measured from the higher anchor, is 20 ft. If the maximum allowable cable tension is 10 kips, state the allowable horizontal load and the necessary length of the cable.



Problem (9)



Solution (9)

Solution (9): $H = \frac{q(L_1)^2}{8d_1} = \frac{q(L_1)^2}{8\times5}$ & $H = \frac{q(L_2)^2}{8d_2} = \frac{q(L_2)^2}{8\times20}$

$\Rightarrow \frac{q(L_1)^2}{8\times5} = \frac{q(L_2)^2}{8\times20}$; $\Rightarrow L_2 = 2L_1$

$\Rightarrow \frac{L_1}{2} + \frac{L_2}{2} = 81$; $\Rightarrow L_1 + L_2 = 162$

$L_1 = 54$ feet & $L_2 = 108$ feet.

$$H = \frac{q(L_1)^2}{8d_1} = \frac{q(54)^2}{8\times5} = 72.9q \text{ lb.}$$

$$\text{or } H = \frac{q(L_2)^2}{8d_2} = \frac{q(108)^2}{8\times20} = 72.9q \text{ lb.}$$

T_{max} occurs at the highest support.

$$T_{max} = T_2 = \sqrt{H^2 + \left(\frac{qL_2}{2}\right)^2} = \sqrt{(72.9q)^2 + \left(\frac{q\times108}{2}\right)^2}$$

$\Rightarrow q = 110.2273242$ lb/ft.

$$S_1 = L_1 + \frac{8d_1^2}{3L_1} - \frac{32d_1^4}{5L_1^3}$$

$$\Rightarrow S_1 = 54 + \frac{8(5)^2}{3(54)} - \frac{32(5)^4}{5(54)^3} = 55.20916527 \text{ feet.}$$

$$S_2 = L_2 + \frac{8d_2^{\ 2}}{3L_2} - \frac{32d_2^{\ 4}}{5L_2^{\ 3}}$$

$$\Rightarrow S_2 = 54 + \frac{8(20)^2}{3(108)} - \frac{32(20)^4}{5(108)^3} = 117.063659 \text{ feet.}$$

$$S = \frac{S_1}{2} + \frac{S_2}{2} = \frac{55.20916527}{2} + \frac{117.063659}{2} = 86.13641213 \text{ feet.}$$

Answer (9): 110.2273242 lb/ft.; 86.13641214 ft.

Problem (10): A clothesline weighs 0.20 lb/ft. it is hung between posts 40 ft apart, and it is 7 ft off the ground at each end. What must be the length of the clothesline and the maximum tension if it is to be hung so that a 6-ft man can walk underneath the lowest point?

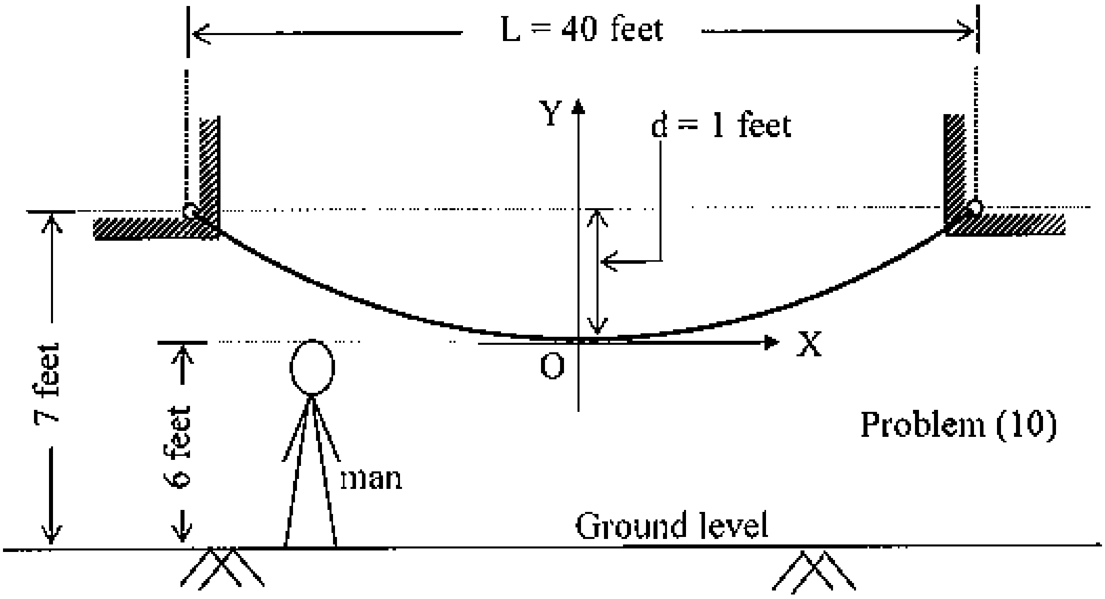
Solution (10): Given, uniform cable weight (q) = 0.20 lb/ft

Span (L) = 40 feet

Sag (d) = 7 feet – 6 feet = 1 feet (see figure below)

$$\frac{\text{sag}}{\text{span}} = \frac{1}{40} = 0.025 = 2.5\%$$

Since, sag is 2.5% of span; the cable can be analyzed as a parabola.



Problem (10)

Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{(0.20)\times(40)^2}{8(1)} = 40 \text{ lb.}$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow T_{max} = \sqrt{(40)^2 + \left(\frac{0.20 \times 40}{2}\right)^2} = 40.19950248 \text{ lb.}$$

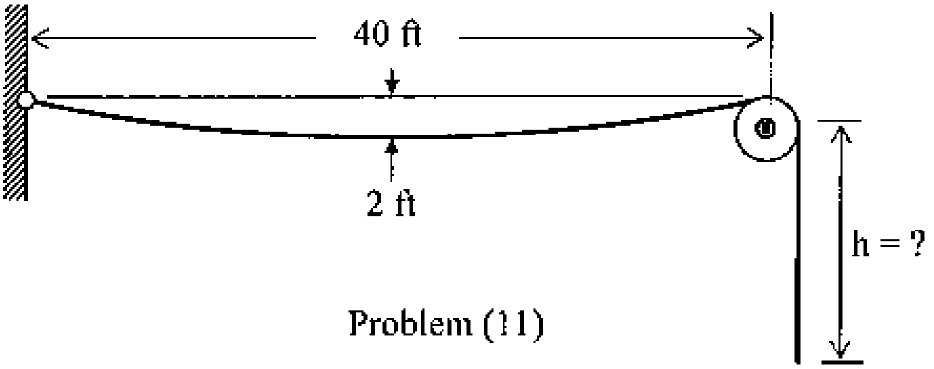
Length of cable = S feet

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 40 + \frac{8(1)^2}{3(40)} - \frac{32(1)^4}{5(40)^3} = 40.06656667 \text{ ft}$$

Answer (10): 40.06656667 ft.; 40.19950248 lb.

Problem (11): A rope weighing 1.5 lb/ft, is anchored to a wall and passed over a frictionless drum, as shown in the figure. What must be the minimum length of the hanging rope to prevent slip on the drum? The co-efficient of friction on the drum is 0.6, and the angle of wrap can be assumed to be π/2 radians.



Problem (11)

Solution (11): Uniform cable weight (q) = 1.5 lb/ft
Span (L) = 40 feet
Sag (d) = 2 feet

$$\frac{\text{sag}}{\text{span}} = \frac{2}{40} = 0.05 = 5\%$$

Since, sag is 5% of span; the cable can be analyzed as a parabola.
Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{1.5 \times 40^2}{8 \times 2} = 150 \text{ lb.}$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow T_{max} = \sqrt{(150)^2 + \left(\frac{1.5 \times 40}{2}\right)^2}$$

$$\Rightarrow T_{max} = 152.9705854 \text{ lb.}$$

Tension on slack side = $T_1 = T_{max}$ = 152.9705854 lb.

Tension on slack side = T_0

Friction co-efficient (μ) = 0.6

Angle of wrap (θ) = $\frac{\pi}{2}$ radian

$$T_1 = T_0 e^{\mu\theta}$$

$$\Rightarrow 152.9705854 = T_0 e^{(0.6 \times \frac{\pi}{2})}$$

$$\Rightarrow T_0 = 59.6066923 \text{ lb.} = (\text{length}) \times (\text{weight per unit length}) = h \times q$$

$$\Rightarrow T_0 = 59.6066923 \text{ lb.} = h \times q = (h \text{ ft}) \times (1.5 \text{ lb/ft})$$

$$\Rightarrow h = 39.73779486 \text{ ft.}$$

Answer (11): h = 39.73779486 ft.

Problem (12): During an ice storm, a cylinder of ice forms on a telephone line. The line is strung between poles 80 ft apart. The weight of the clean line is 0.3 lb/ft. How much ice can form if the sag is not to exceed 5 ft and the maximum allowable tension in the line is 600 lb. Assume the ice is to be a solid cylinder with wet density 56 lb/cft.

Solution (12): Span (L) = 80 feet
Sag (d) = 5 feet

$$\frac{\text{sag}}{\text{span}} = \frac{5}{80} = 0.0625 = 6.25\%$$

Since, sag is 6.25% of span; the cable can be analyzed as a parabola.
Weight of clean line = 0.3 lb/ft
Weight of ice formed = w lb/ft

Total uniform weight on the cable (ice weight + cable self-weight) = q = (0.3 + w) lb/ft
Maximum tension (T_{max}) = 600 lb.

Minimum tension at lowest point = H

$$H = \frac{(0.3+w)\times(80)^2}{8\times5}$$

$$\Rightarrow H = 160\ (0.3 + w)$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow\ 600 = \sqrt{[160(0.3+w)]^2 + \left(\frac{(0.3+w)\times(80)}{2}\right)^2}$$

$$\Rightarrow w = 3.338034376 \text{ lb/ft} = \text{weight of ice-formed}$$

Ice unit weight = 56 lb/cft

$$\text{Cross-sectional area of ice} = \frac{3.33803437\ 6 \text{ lb/ft}}{56 \text{ lb/cft}} = 0.059607756\ \text{ft}^2$$

$$= 0.059607756 \times 144 \text{ inch}^2 = 8.583516966 \text{ inch}^2$$

$$\Rightarrow\ \frac{\pi(D)^2}{4} = 8.583516966 \text{ inch}^2$$

$$\Rightarrow D = 3.305884637 \text{ inch}$$

Answer (12): 3.305884637 inch diameter ice cylinder.

Problem (13): A 70-ft cable, weighing 7 lb/ft, hangs between two supports at the same elevation. The sag is 20 ft. how far apart must the supports be placed.

Solution (13): Total length of the cable (2S) = 70 feet
S = 35 feet
Uniform load along the cable (q) = 7 lb/ft
Sag (d) = 20 feet
Span of cable = L feet

$$y = k + d = \sqrt{k^2 + s^2}$$

$$\Rightarrow k + 20 = \sqrt{k^2 + 35^2}$$

$$\Rightarrow\ (k+20)^2 = k^2 + 35^2$$

$$\Rightarrow\ k^2 + 40k + 400 = k^2 + 1225$$

$$\Rightarrow k = \frac{1225 - 400}{40} = 20.625 \text{ feet}$$

$$s = k\sinh\left(\frac{x}{k}\right)$$

at x = L/2, S = 35 feet

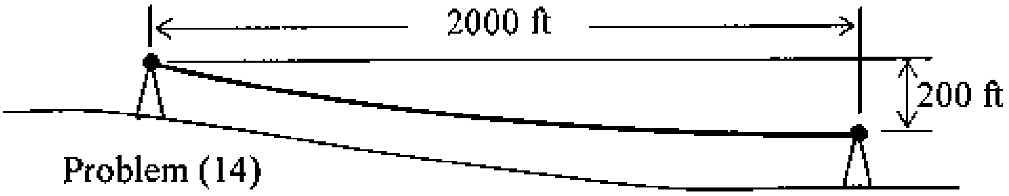
$$\Rightarrow\ 35 = 20.625\sinh\left(\frac{(L/2)}{20.625}\right)$$

$$\Rightarrow\ \sinh\left(\frac{L}{41.25}\right) = 1.696969697$$

$$\Rightarrow L = 41.25\sinh^{-1}(1.696969697) = 53.5954231 \text{ feet}$$

Answer (13): 53.5954231 feet

Problem (14): An overland transmission line cable is strung between 80-ft towers 2000 ft apart. The difference in elevation of the towers is 200 ft. the cable weighs 4 lb/ft. the minimum tension (H) is known to be 40,000 lb. what is the tension in the cable at the towers? What will be the length of the cable?



Problem (14)

Solution (14): Uniform cable weight (q) = 4 lb/ft
Minimum tension at lowest point (H) = 40000 lb.
Sag (d) = 200 ft
Distance between tower = 2000 ft = half of span
Span (L) =2 × 2000 ft = 4000 ft

$$\frac{\text{sag}}{\text{span}} = \frac{200}{4000} = 0.050 = 5\%$$

Since, sag is 5% of span; the cable can be analyzed as a parabola.

$$T_{\max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow\ T_{\max} = \sqrt{(40000)^2 + \left(\frac{4\times 4000}{2}\right)^2} = 40{,}792.15611 \text{ lb.}$$

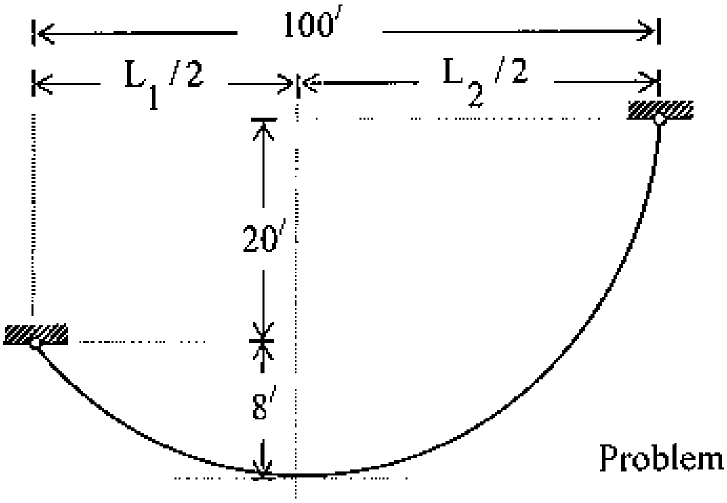
$$S_{\max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow\ S_{\max} = 4000 + \frac{8(200)^2}{3(4000)} - \frac{32(200)^4}{5(4000)^3} = 4026.506667 \text{ ft}$$

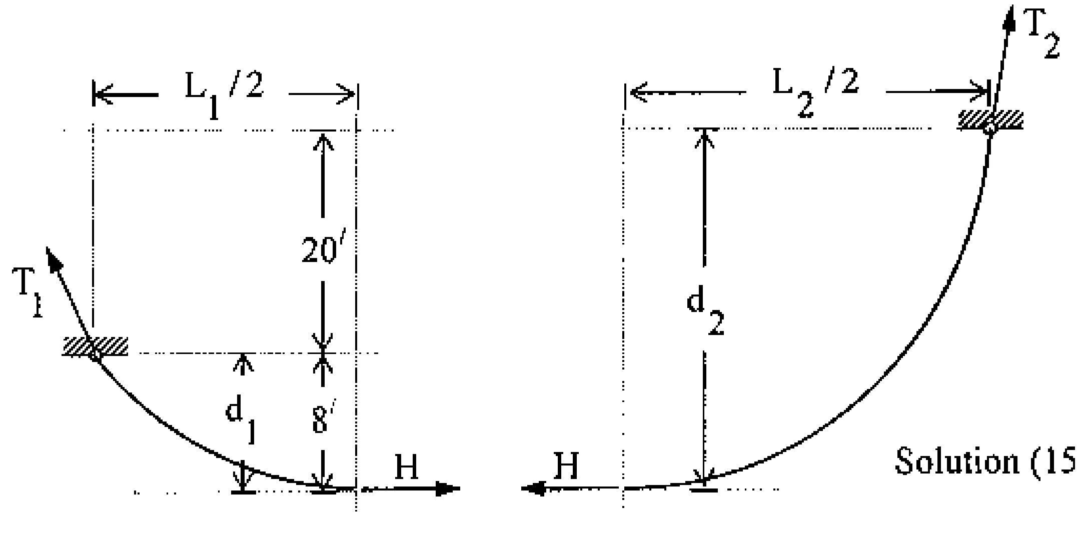
Length of cable (S/2) = 2013.253333 ft

Answer (14): 40,792.15611 lb.; 2013.253333 ft

Problem (15): a parabolic cable carries a load of 200 lb per horizontal foot supported at two ends 100 ft apart & one support is 20 ft higher than the other. The sag measured from the lower support is 8 ft. Determine the tension at the anchors.



Problem (15)



Solution (15)

Solution (15): $H = \frac{q(L_1)^2}{8d_1} = \frac{200(L_1)^2}{8\times 8}$ & $H = \frac{q(L_2)^2}{8d_2} = \frac{200(L_2)^2}{8\times 28}$

$$\Rightarrow \frac{200(L_1)^2}{8\times 8} = \frac{200(L_2)^2}{8\times 28}; \Rightarrow L_2 = \sqrt{\frac{7}{2}}\,L_1$$

$$\Rightarrow \frac{L_1}{2} + \frac{L_2}{2} = 100; \Rightarrow L_1 + L_2 = 200$$

$L_1 = 69.66629547$ feet & $L_2 = 130.3337045$ feet.

$$H = \frac{q(L_1)^2}{8d_1} = \frac{200(69.66629547)^2}{8\times 8} = 15166.85226 \text{ lb.}$$

$$\text{or } H = \frac{q(L_2)^2}{8d_2} = \frac{200(130.3337045)^2}{8\times 28} = 15166.85226 \text{ lb.}$$

$$T_1 = \sqrt{H^2 + \left(\frac{qL_1}{2}\right)^2} = \sqrt{(15166.85226)^2 + \left(\frac{200\times 69.66629547}{2}\right)^2}$$

$$\Rightarrow T_1 = 16690.33657 \text{ lb.}$$

$$T_2 = \sqrt{H^2 + \left(\frac{qL_2}{2}\right)^2} = \sqrt{(15166.85226)^2 + \left(\frac{200\times 130.3337045}{2}\right)^2}$$

$$\Rightarrow T_2 = 19997.55367 \text{ lb.}$$

Answer (15): 16,690.33657 lb & 19,997.55367 lb.

Problem (16): A transmission-line cable is to be hung from two towers, at the same elevation and 800 ft apart. The weight of the cable is 5 lb/ft, and the length of the cable is 1000 ft. What are the necessary sag and maximum tension?

Solution (16): Total length of the cable (2S) = 1000 feet
S = 500 feet
Uniform load along the cable (q) = 5 lb/ft
Span of cable (L) = 800 feet
Sag = d
Maximum tension = T_{max}

$$s = k\sinh\left(\frac{x}{k}\right)$$

at x = L/2 = 400 feet, S = 500 feet

$$\Rightarrow 500 = k\sinh\left(\frac{400}{k}\right)$$

Solving, by trial-&-error, k = 338.2096278 feet

$$y = k + d = \sqrt{k^2 + s^2}$$

$$\Rightarrow 338.2096278 + d = \sqrt{338.2096278^{\ 2} + 500^2}$$

$$\Rightarrow d = 265.4427328 \text{ feet}$$

Maximum tension (which occurs at the supports (at x = L/2 & y = k+d)) is

$T_{max} = q(k+d) = 5(338.2096278 + 265.4427328) = 3018.163664$ lb.

Answer (16): 265.4427328 ft & 3018.163664 lb.

Problem (17): A bridge cable is to be swung between two towers 1200 ft apart. The strength of the cable is 1000 kips. The loading on the cable is 400 lb per horizontal foot. Determine the necessary sag and the total length of the cable. The ends of the cable are at the same elevation.

Solution (17): Uniform loading on the cable (q) = 400 lb per horizontal ft
Loading is given per horizontal feet; analyze the cable as a parabola.
Span (L) = 1200 feet
Sag = d
Maximum tension (T_{max}) = 1000 kip = 1000000 lb.

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 1000000 = \sqrt{H^2 + \left(\frac{400 \times 1200}{2}\right)^2}$$

$$\Rightarrow H = 970772.888 \text{ lb.} = \text{Minimum tension at lowest point}$$

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow 970772.888 = \frac{(400)\times(1200)^2}{8(d)}$$

$$\Rightarrow d = 74.16770791 \text{ feet} = \text{Sag}$$

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 1200 + \frac{8(74.16770791)^2}{3(1200)} - \frac{32(74.16770791)^4}{5(1200)^3}$$

$$\Rightarrow S_{max} = 1212.112037 \text{ feet} = \text{Length of cable}$$

Answer (17): 74.16770791 ft; 1212.112037 ft

Problem (18): A 60-ft cable weighs 360 lb. and is suspended from two points at the same elevation. For a sag of 15 ft., find (a) the tension at the supports & (b) the span.

Solution (18): Total length of the cable (2S) = 60 feet

S = 30 feet

Uniform load along the cable (q) = $\frac{\text{weight}}{\text{length}} = \frac{360\text{ lb}}{60\text{ ft}} = 6$ lb/ft

Sag (d) = 15 feet
Span of cable = L feet
Maximum tension = T_{max}

$$y = k + d = \sqrt{k^2 + s^2}$$
$$\Rightarrow k + 15 = \sqrt{k^2 + 30^2}$$
$$\Rightarrow (k+15)^2 = k^2 + 30^2$$
$$\Rightarrow k^2 + 30k + 225 = k^2 + 900$$
$$\Rightarrow k = \frac{900 - 225}{30} = 22.5 \text{ feet}$$

Maximum tension (which occurs at the supports (at x = L/2 & y = k+d)) is
$T_{max} = q\,(k + d) = 6\,(22.5 + 15) = 225$ lb.

$$s = k\sinh\left(\frac{x}{k}\right)$$

at x = L/2, S = 30 feet

$$\Rightarrow 30 = 22.5\sinh\left(\frac{(L/2)}{22.5}\right)$$

$$\Rightarrow \sinh\left(\frac{L}{45}\right) = 7.5$$

$$\Rightarrow L = 45\sinh^{-1}(7.5) = 49.43755299 \text{ feet} = \text{Span of cable}$$

Answer (18): (a) 225 lb.; (b) 49.43755299 ft.

Problem (19): A cable weighing 1 lb. per ft. is suspended between two points on the same level with a span of 300 ft. and sag of 20 ft. What are (a) the length of cable, (b) the tension at the lowest point, & (c) the maximum tension?

Solution (19): Uniform cable weight (q) = 1 lb/ft
Span (L) = 300 feet
Sag (d) = 20 feet

$$\frac{\text{sag}}{\text{span}} = \frac{20}{300} = 0.06667 = 6.667\%$$

Since, sag is 6.667% of span; the cable can be analyzed as a parabola.

Minimum tension at lowest point = H

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow H = \frac{(1)\times(300)^2}{8(20)} = 562.5 \text{ lb.}$$

Maximum tension (which occurs at the supports (at x = L/2, y = d)) is

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow T_{max} = \sqrt{(562.5)^2 + \left(\frac{1\times300}{2}\right)^2} = 582.1565511 \text{ lb.}$$

Length of cable = S feet

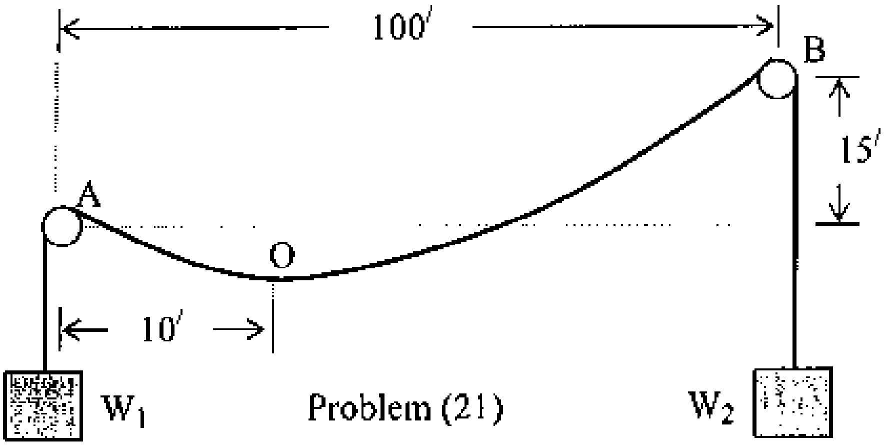
$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 300 + \frac{8(20)^2}{3(300)} - \frac{32(20)^4}{5(300)^3} = 303.5176296 \text{ ft}$$

Answer (19): (a) 303.5176296 ft.; (b) 56.5 lb.; (c) 582.1565511 lb.

Problem (20): A wire weighing 0.25 lb per ft. is strung between two supports, 300 ft. apart. One support is 15 ft. higher than the other, and the sag, measured from the lowest support, is 5 ft. Compute the tension at each support and the length of wire.
Hints: See solution (9) & solution (15)
Answer (20): 251.2468905 lb., 254.9509757 lb. & 301.49175 ft.

Problem (21): A cable weighs 2 lb. per ft. supported at two ends with one support 15 ft. higher than the other. Consider initially that the cable is parabolic and that negligible lengths of cable hang vertically. Find the weights W_1 & W_2 for equilibrium. Is the sag to span ratio such as to justify recalculation as a catenary? A & B are smooth pegs of negligible diameter and O is the lowest point in the span.
Answer (21): $W_1 = 533.7082016$ lb. & $W_2 = 562.8893714$ lb.



Problem (21)

Problem (22): A power line which weighs 0.25 lb. per ft. is to be subjected to a maximum tension of 1500 lb. The poles are 200 ft. apart. If the supports are in a horizontal plane and if the weight is assumed to be uniformly distributed horizontally, what is the corrected sag? What is the length of the wire between poles?

Solution (22): Uniform loading on the cable (q) = 0.25 lb per horizontal ft
Loading is given per horizontal feet; analyze the cable as a parabola.
Span (L) = 200 feet
Sag = d
Maximum tension (T_{max}) = 1500 lb.

$$T_{max} = \sqrt{H^2 + \left(\frac{qL}{2}\right)^2}$$

$$\Rightarrow 1500 = \sqrt{H^2 + \left(\frac{0.25 \times 200}{2}\right)^2}$$

$$\Rightarrow H = 1499.791652 \text{ lb.} = \text{Minimum tension at lowest point}$$

$$H = \frac{qL^2}{8d}$$

$$\Rightarrow 1499.791652 = \frac{(0.25) \times (200)^2}{8(d)}$$

$$\Rightarrow d = 0.833449098 \text{ feet} = \text{Sag}$$

$$S_{max} = L + \frac{8d^2}{3L} - \frac{32d^4}{5L^3}$$

$$\Rightarrow S_{max} = 200 + \frac{8(0.833449098)^2}{3(200)} - \frac{32(0.833449098)^4}{5(200)^3}$$

$$\Rightarrow S_{max} = 200.0092614 \text{ ft} = \text{Length of cable}$$

Answer (22): 0.833449098 ft.; 200.0092614 ft.

Problem (23): A suspension foot bridge is supported by two parallel cables, one on each side of the bridge. The suspension points are in a horizontal plane. The span is 300 ft., the sag is 40 ft., and the cables each weigh 3 lb. per ft. the floor of the bridge is 4 ft. wide and weighs 30 lb. per ft. of length. For a maximum permissible tension in a cable of 20000 lb., what live load per square foot of bridge-floor area can this bridge sustain? Neglecting stretch due to load, find the length of each cable.
Answer (23): 14.01571559 lb. per square foot; 313.7776029 ft.

Problem (24): A cable weighing 5 lb. per ft. hangs as a catenary from points which are 600 ft. apart and on the same level. The tension at the lowest point is 1500 lb. Find (a) the length of the cable, (b) the sag, & (c) the maximum tension in the cable. Solve this problem assuming that then cable forms a parabola and compare answers.
Answer (24):

	Catenary	Parabola
Length of the cable	705.1207162 ft	685 ft
Sag	162.9241904 ft	150 ft
Maximum tension	2314.620952 lb	1507.481343 lb

Problem (25): If a cable, 200 ft. long and weighing 3 lb. per ft., is suspended from points on a horizontal with a span of 150 ft., find (a) the minimum tension, (b) the maximum tension, (c) the sag.

Solution (25): Total length of the cable (2S) = 200 feet
S = 100 feet
Uniform load along the cable (q) = 3 lb/ft
Span of cable (L) = 150 feet
Sag = d
Maximum tension = T_{max}

$$s = k\sinh\left(\frac{x}{k}\right)$$

at x = L/2 = 75 feet, S = 100 feet

$$\Rightarrow 100 = k\sinh\left(\frac{75}{k}\right)$$

Solving, by trial-&-error, k = 55.50 feet

$$y = k + d = \sqrt{k^2 + s^2}$$

$\Rightarrow 55.50 + d = \sqrt{55.5^2 + 100^2}$
$\Rightarrow d = 58.8689206$ feet

Minimum tension (which occurs at lowest point), $H = q\,k = 5\,(55.5) = 166.5$ lb.
Maximum tension (which occurs at the supports (at $x = L/2$ & $y = k+d$)) is
$T_{max} = q\,(k+d) = 5\,(55.5 + 58.8689206) = 343.1067918$ lb.

Answer (25): (a) 166.50 lb.; (b) 343.1067918 lb.; (c) 58.8689206 ft.

Problem (26): A cable in the form of a catenary is 400 ft. long. How far apart may be the supports on the same level, if the maximum tension is not to exceed 400w, where w is the weight of the cable in pounds per foot?

Solution (26): Given, Maximum tension, $T_{max} = 400\,w$
total length of the cable (2S) = 400 feet
S = 200 feet
Uniform load along the cable (q) = w lb/ft
Span of cable = L feet

Maximum tension (which occurs at the supports (at $x = L/2$ & $y = k+d$)) is

$$T_{max} = w\,(k+d) = w\,(k+d) = 400\,w$$
$$\Rightarrow (k+d) = 400$$

$$y = k + d = \sqrt{k^2 + s^2}$$
$$\Rightarrow 400 = \sqrt{k^2 + 200^2}$$
$$\Rightarrow k = 346.4101614 \text{ feet}$$

$$s = k\sinh\left(\frac{x}{k}\right)$$

at $x = L/2$, $S = 200$ feet

$$200 = 346.4101614\sinh\left(\frac{(L/2)}{346.4101614}\right)$$

$$\sinh\left(\frac{L}{692.820323}\right) = 0.577350269$$

$$\Rightarrow L = 692.820323\sinh^{-1}(0.577350269) = 380.5704604 \text{ ft.}$$

Answer (26): 380.5704604 ft.